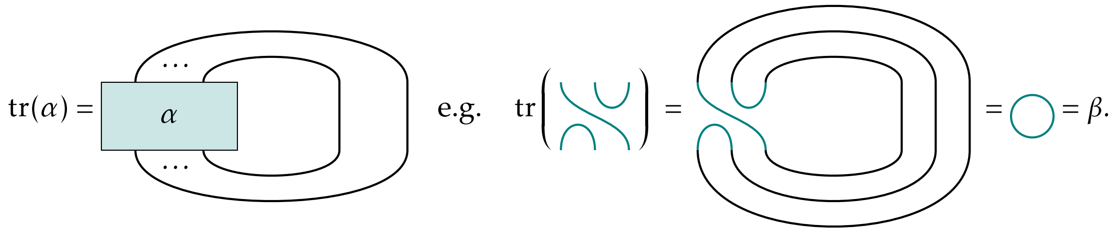
Oral examination preparation, 21-07-2025

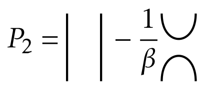
This sheet gives you an idea of the type of questions that will be asked during your oral examination. These will **not** be the questions asked at the oral examination. In addition, they do not include the follow-up questions I will ask. That being said, they are similar to the questions and should help you have a clear idea of what can be asked. The examination should take 30 minutes. I have added in parenthesis the time in minute I expect you to spend on each questions.

You do not need to give the "correct" answer to obtain full grade. You do need to explain your thought process.

1. A diagrammatic computation (2) We define a "trace" on the Temperley–Lieb algebra $\mathsf{TL}_n(\beta)$ that we defined in the course. It is a function $\mathrm{tr} : \mathsf{TL}_n(\beta) \to \mathbb{C}$ given by the following diagrammatic construction: given a Temperley–Lieb diagram α, embed it into a bigger space and connect the top and bottom strands with loops via the diagrammatic rule:



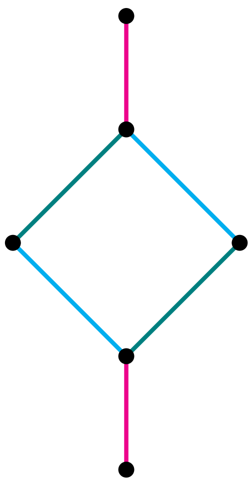
Compute the trace of the following element of $\mathsf{TL}_2(\beta)$

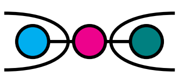


(1)

2. Cellular algebras (5) How do you define the simple modules of a given cellular algebra? Can you give one example from the course?

3. Paths on the Bruhat graph for the symmetric group with maximal parabolic (5) Consider the following Bruhat graph of the maximal parabolic system $(\mathfrak{S}_4, \mathfrak{S}_2 \times \mathfrak{S}_2)$ with Coxeter generators $S^W = \{\tau, \sigma, \rho\}$ and parabolic $P = \langle \tau, \rho \rangle$.





1. Write down the following paths of colour pattern $\sigma\rho\tau\sigma$

$$S_1 = U_2^1 U_3^1 U_1^1 U_2^1,$$
$$S_2 = U_2^1 U_3^1 U_1^1 U_2^0,$$
$$S_3 = U_2^1 U_3^0 U_1^0 D_2^0,$$
$$S_4 = U_2^1 U_3^0 U_1^0 D_2^1,$$

and compute their degree.

2. Can you draw the Soergel diagram d_{S_3} of the algebra $\mathcal{H}_{\mathfrak{S}_4, \mathfrak{S}_2 \times \mathfrak{S}_2}$? If you do not remember the procedure, describe the shape of the diagram (what would go on top, what on the bottom, and what is inside), then go check your course notes and draw it.

4. Correspondence between diagrammatic and representation theory (8) Consider the subalgebra $\mathsf{TL}_2(\beta) \subset \mathsf{TL}_3(\beta)$. We look back at the elements $P_2 \in \mathsf{TL}_2(\beta)$ defined in (1) and its inclusion in $\mathsf{TL}_2(\beta)$.

1. Compute the action of the generator of $\mathsf{TL}_2(\beta)$ on $P_2 \in \mathsf{TL}_2(\beta)$ defined in (1).

   Do you have an interpretation of this computation in representation-theoretical terms?

2. Compute the actions of the generators of $\mathsf{TL}_3(\beta)$ on $P_2 \in \mathsf{TL}_2(\beta)$ under its inclusion in $\mathsf{TL}_3(\beta)$.

   Can you say something about the results of this computation?

5. A new algebra (hard) (10) Consider the tower of inclusion $\mathsf{TL}_2(\beta) \subset \mathsf{TL}_3(\beta) \subset \cdots \subset \mathsf{TL}_{n+2}(\beta)$ for $n \geq 0$. We denote $P_2^n \in \mathsf{TL}_{n+2}(\beta)$, the image of P_2 defined in (1) under these inclusion. We consider the idempotent truncation algebra $B_{n,2}(\beta) := P_2^n \mathsf{TL}_{n+2}(\beta) P_2^n$. We call it the boundary seam algebra[1]

1. Do you think this algebra is well defined?
2. Do you have a proposal for a diagrammatic calculus?
3. Do you think this algebra is semisimple?
4. If you were to characterise its simple modules, how would you go about it?

Now is a good time to reiterate that I do not expect you to blitz out all the correct answers, nor the complete steps of a solution right away to obtain full grade. The goal is to tell me how you would do and ask what you need. For example, a full answer to question 3.1 can be: "Here are the answers for S_1, S_2, but I don't remember if D_i^0 is going down or failing to go down, so the two last paths are uncertain. For their degrees, I remember reduced paths should be of degree zero so the U_i^1's need to be degree zero, but I'm not sure of the others; I will assume this and then the full degree is the sum of the degrees of the U and D."

References

[MRR15] A. Morin-Duchesne, J. Rasmussen, and D. Ridout. "Boundary algebras and Kac modules for logarithmic minimal models". In: *Nuclear Physics B* 899 (2015), pp. 677–769. arXiv: 1503.07584.

[1] The name comes from physics [MRR15].