Problem sheet 7, 02-06-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de or by accessing online the book.

1. Compute
$$\begin{bmatrix} n+m \\ m \end{bmatrix}_q$$
 for $(m,n) = (2,2), (3,2)$ and $(3,3)$

2. Construct the Bruhat graph for (m, n) tile-partitions for the (m, n) above and compare with the value of the coefficients you found previously. (Note that Figs 7.2, 7.9 give you some indication.)

1. Paths in the Bruhat graph Compute all the potential paths in $\mathcal{P}_{2,3}$ of fixed colour sequence $s_2s_3s_4s_1s_2s_3$ and draw the path in the graph $\mathcal{P}_{2,3}$. (We did two of them in class.) Then, associate the Kazdhan–Lusztig polynomial to them and their degree. Remember, the moves for $\lambda \xrightarrow{s_i} \mu$ are

$U_i^1: \lambda \to \mu$,	$\deg U_{i}^{1} = 0;$
$U_i^0: \lambda \to \lambda$,	$\deg U_i^0 = 1;$
$D_i^1:\mu \to \lambda$,	$\deg D_i^1 = 0;$
$D_i^0:\mu \to \mu$,	$\deg D_i^0 = -1.$

2. Kazhdan–Lusztig polynomials for $\mathfrak{S}_2 \times \mathfrak{S}_2 \leq \mathfrak{S}_4$ Compute all the Kazhdan–Lusztig polynomials for $\mathfrak{S}_2 \times \mathfrak{S}_2 \leq \mathfrak{S}_4$ and draw their path on the Bruhat graph. (See Fig 7.5 in the book for the answer.)

3. (Exo 7.3.6) Kazhdan–Lusztig polynomials for $\mathfrak{S}_2 \times \mathfrak{S}_3 \leq \mathfrak{S}_5$ Compute all the Kazhdan–Lusztig polynomials for $\mathfrak{S}_2 \times \mathfrak{S}_3 \leq \mathfrak{S}_5$ and draw their path on the Bruhat graph. (See Fig 7.2 in the book for the graph and Exercise 1 of this sheet, or Example 7.3.2 of the book for the first column (the $n(\lambda, \mu)$ for maximal colour sequence).) This is a 10×10 matrix.

4. Complete the isomorphism We swept under the carpet that the two ways of seeing the poset (with weight diagrams and tile-partitions) were the same. The beginning is done in Proposition 7.4.2 of the book; complete the details.