Problem sheet 6, 19-05-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de or by accessing online the book.

0. (Drill)

1. The algebra of polynomial of n variables $A = \mathbb{Z}[x_1, \dots, x_n]$ is a $\mathbb{Z}$–graded algebra (in fact, a $\mathbb{N}$–graded algebra). What are the level A_i in the decomposition $A = \bigoplus_{i\in\mathbb{Z}} A_i$?
2. Compute some q-binomials (pay attention to the typo in the definition of the q-numbers). They are defined as
$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}$$
where $[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q!$ with $[0]_q := 1$.

1. The zig-zag algebra Read Section 5.4 on the zig-zag algebra and complete the exercises.

2. Idempotent and grading tricks in action Read Section 6.8 regarding the simple modules of the zig-zag algebra to appreciate the use of the grading and idempotent tricks. Then rewrite the section in as much details as you want for a small zig-zag algebra to convince yourself.

3. Relating the binary Schur and the zig-zag algebra We have seen in Lecture 5 how some idempotent truncation algebra can still be cellular. In particular, the zig-zag algebra ZZ_2 for example is an idempotent truncation of the binary Schur algebra, that is, $\mathsf{ZZ}_2 = e_\Pi S_{\mathbb{F}_2}(2)e_\Pi$ for some idempotent fixed the anti-involution e_Π. Give more details than Example 6.12.2, and extend this observation to ZZ_3 (pay attention as the zig-zag algebra field changes).

4. Understanding some poset construction Read Section 7.2 and convince yourself of the combinatorics you see there. This is best done by doing everything for the parabolic (if you forgive me for starting earlier with name) $\mathfrak{S}_2 \times \mathfrak{S}_2 \leq \mathfrak{S}_4$ and, in particular, Figure 7.2.