Lecture notes and Problem sheet 4, 05-05-2025

Extra material and preliminaries of representation theory of algebra

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de

Some refresher of finite-dimensional algebras over fields

This is a small compendium of results for finite-dimensional algebras. In general, we will not need that much so they will only be stated here without proof. References for this are the appendix of the book of Mathas [Mat99] and the classic of Curtis–Reiner [CR66].

Let A be a finite-dimensional algebra over a field $\mathbb{F}$. We will assume everything is over $\mathbb{C}$ to not trouble ourselves. Let M be a finite-dimensional A-module. We say M is a *simple* (or *irreducible*) if M is a proper and has no non-trivial proposer submodule. A *filtration* of M is a sequence of A-submodules of A

$$0 = M_0 \subset M_1 \subset M_1 \subset \cdots \subset M_k \subset M_{k+1} = M.$$

A *composition series* of M is a filtration of M where each *composition factor* M_i/M_{i-1} is a simple A-module.

Lemma 1. *Every A-module M has a composition series.*

Proof. Induction on the dimension of M. □

In particular, A viewed as an A-module on itself also has a composition series.

Lemma 2. *Suppose L is a simple A-module. Then $L \simeq A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ and L is a composition factor of the A-module A.*

One of the key results in the representation theory of finite-dimensional algebras is that those composition series are well-defined and that, even though you will be able to filter a module in many ways, its composition factors are unique up to reordering.

Theorem 3 (Jordan–Hölder)**.** *Suppose that M is an A-module and that*

$$0 = M_1 \subset \cdots \subset M_k \subset M_{k+1} = M, \qquad 0 = N_1 \subset \cdots \subset N_l \subset N_{l+1} = M$$

are two composition series of M. Then $k = l$ and for each simple module A-module L

$$|\{i \mid L \simeq M_{i+1}/M_i\}| = |\{i \mid L \simeq N_{i+1}/N_i\}|.$$

Remark 4. *The Jordan–Hölder Theorem does not hold for $\mathbb{Z}$-modules.*

In particular, the Jordan–Hölder Theorem lets us define the *composition multiplicities* $[M : L]$ of L in M as the number of composition factors of M that are isomorphic to L.

Extra material on the course

Since I departed slightly from Chris's convention for cellularity (and also used left-module to keep in line with the general notion), I will provide some course notes. Feel free to follow Chris' convention instead of mine. They are slightly less general, but most cellular algebras of relevance fall in Chris' convention. I follow here mostly the original paper of Graham and Lehrer [GL96] with some proofs taken from the book of Mathas [Mat99]. Whatchout if you are to look into the book of Mathas: he considers right-modules and his order is the reverse of the other sources (and mine in particular).

We begin with the original definition of cellular algebras due to Graham–Lehrer. Here, we put R a commutative ring with unit and our algebra are associative and unital.

Definition 5 ([GL96, Definition 1.1]). *A* cellular algebra *is an R-algebra together with a* cell datum $(\Lambda, P, C, *)$ *where*

- $\Lambda = (\Lambda, \preceq)$ *is a poset;*
- *for each $\lambda \in \Lambda$, $P(\lambda)$ is a finite set;*
- $C : \bigsqcup_{\lambda\in\Lambda} P(\lambda)\times P(\lambda) \to A$ *is an injective map whose image is an R-basis of A. We write $C(B,T) = C^\lambda_{BT}$ for $B,T \in P(\lambda)$. This basis is called the* cellular basis *of A;*
- $* : A \to A$ *is an anti-involution,*

and the following relations are respected

$$aC^\lambda_{BT} \equiv \sum_{S\in P(\lambda)} r^a_{SB} C^\lambda_{ST} \mod A^{<\lambda} \tag{1}$$

$$(C^\lambda_{BT})^* = C^\lambda_{TB}, \tag{2}$$

where $A^{<\lambda}$ is generated by $\{C^\mu_{B'T'} \mid \mu < \lambda, B', T' \in P(\mu)\}$.

It will be useful to combine (1) and (2):

$$C^\lambda_{BT} a \equiv \sum_{U\in P(\lambda)} r^a_{BU} C^\lambda_{BU} \mod A^{<\lambda} \tag{3}$$

The whole point of the definition is that this gives us, on the nose, a family of modules coming from the $P(\lambda)$. We fix for the next part, a cellular algebra over a field $R(=\mathbb{C})$ with cell datum $(\Lambda, P, C, *)$.

Definition 6. *We define the (left)* cell module V^λ *for $\lambda \in \Lambda$ to be the free vector space with basis $\{v_B \mid B \in P(\lambda)\}$ and action given by*

$$av_B = \sum_{S\in P(\lambda)} r^a_{BB'} v_{B'}$$

where $r^a_{BB'}$ is determined by aC^λ_{BB}.

More precisely, we define V^λ_T as the R-submodule of $A^{\leq\lambda}/A^{<\lambda}$ with basis $\{C^\lambda_{BT} + A^{<\lambda} \mid B \in P(\lambda)\}$. It is a left A-module by (1), and furthermore it is independent of T so we can identify it with V^λ.

Observe that we can use the anti-involution to define right A-modules.

We now want to define the bilinear form. We will use a technical lemma to make sure it is well-defined.

Lemma 7. *Suppose $B, T \in P(\lambda)$. Then there exists an elements $r_{BT} \in R$ such that, for any $S, U \in P(\lambda)$*

$$C^\lambda_{ST} C^\lambda_{BU} \equiv r_{BT} C^\lambda_{SU} \mod A^{<\lambda}.$$

Proof. We simply simplify the product in two ways, first with (1) and then with (3); what will remain is simply one coefficients r_{BT}. □

This allows us to give a bilinear form $\langle -,- \rangle_\lambda$.

Definition 8. *Define a bilinear form $\langle -,- \rangle_\lambda : V^\lambda \times V^\lambda \to R$ on the basis of V^λ by*

$$\langle v_B, v_T \rangle_\lambda = r_{BT} \mod A^{<\lambda},$$

and extending linearly.

The form is symmetric and associative.

Proposition 9. *Let $\lambda \in \Lambda$, $B, T \in P(\lambda)$, $a \in A$ and $v, w \in V^\lambda$.*

1. $\langle v, w\rangle_\lambda = \langle w, v\rangle_\lambda$.
2. $\langle av, w\rangle_\lambda = \langle v, a^* w\rangle_\lambda$.
3. $C^\lambda_{BT} v = \langle v, v_T\rangle_\lambda v_B$.

Proof. Since everything can be done on the basis $v = v_S, w = v_U$ and extended linearly, i) follows easily by application of $*$; ii) follows from the definition of the bilinear form by choosing the parentheses: $\langle av_S, v_U\rangle_\lambda C^\lambda_{UV} \equiv aC^\lambda_{SS}C^\lambda_{UU} \equiv (C^\lambda_{SS}a^*)C^\lambda_{UU} \equiv C^\lambda_{SS}(a^*C^\lambda_{UU}) \equiv \langle v_S, v_U\rangle_\lambda C^\lambda_{SU}$. Finally, the last is precisely the definition of the bilinear form. □

Lemma 10. *Suppose $v \in V^\lambda$ and $a \in A^{\preceq\mu}$. Then $\lambda \prec \mu \Rightarrow av = 0$.*

Proof. Apply iii) of the previous proposition. □

Definition 11. *Let $\mathrm{Rad}_\lambda = \{v \in V^\lambda \mid \langle v, w\rangle_\lambda = 0, \forall w \in V^\lambda\}$ be the radical of the bilinear form. Denote also $L^\lambda := V^\lambda/\mathrm{Rad}_\lambda$.*

We denote $\Lambda^0 = \{\lambda \in \Lambda \mid \mathrm{Rad}_\lambda \neq V^\lambda\}$.

Proposition 12. *Let R be a field[1]; then Rad_λ is the unique maximal submodule of V^λ and L^λ is simple.*

Proof. In class. □

Proposition 13. *Let R be a field and let $\lambda, \mu \in \Lambda^0$. Let M be a proposer submodule of V^μ and suppose that $\sigma : V^\lambda \to V^\mu/M$ is an A-modules morphism.*

1. *If $\sigma \neq 0$ then $\mu \preceq \lambda$.*
2. *If $\mu = \lambda$ then $\sigma(v) = M + r_\sigma v$ for all $v \in V^\lambda$.*

Proof. Use the strategy of the proof of Proposition 12 to use that there exists elements $v, w \in V^\lambda$ such that $\langle v, w\rangle = 1$ and then use v to generate all $v_B \in V^\lambda$. So for v_B we get $\sigma(v_B) = \sigma(a_B v) = a_B\sigma(v) + M$ and then $\lambda \prec \mu$ implies $a_B\sigma(v) = 0$ by Lemma 10. If $\lambda = \mu$, then express $a_B = \sum_{U\in P(\lambda)} r_U C_{BT}$ and thus Proposition 12 iii) implies $\sigma(v_B) = \sigma(v)a_B + M = \sum_{U\in P(\lambda)} r_U \sigma(v) C_{BU}\sigma(v) = \langle\sigma(v), y\rangle v_B$ for $y = \sum_{U\in P(\lambda)} r_U v_U$, and $r_\sigma = \langle\sigma(v), y\rangle_\lambda$. □

Corollary 14. *The L^λ are pairwise non-isomorphic.*

Proof. In class. □

We now prove a technical lemma giving a filtration of the cellular algebra.

Lemma 15. *Suppose that Λ is finite with $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Lambda$ is a maximal chains of ideal of Λ. Then we can find a total ordering $\lambda_1, \ldots, \lambda_k$ such that $\Gamma_i = \{\lambda_1, \ldots, \lambda_i\}$. Then*

$$0 = A(\Gamma_0) \hookrightarrow A(\Gamma_1) \hookrightarrow \cdots \hookrightarrow A(\Gamma_k) = A$$

is a filtration of A with composition factors $A(\Gamma_i)/A(\Gamma_{i-1}) \simeq V^{\lambda_i *} \otimes_R V^{\lambda_i}$.

Proof. Since the chain of ideal is maximal, that means $\Gamma_i \setminus \Gamma_{i-1} = \{\mu\}$ for a certain μ. So we can find a total ordering. As a consequence, $A^{<\lambda_i} \subset A(\Gamma_{i-1})$ and the basis $\{C^{\lambda_i}_{BT} + A(\Gamma_{i+1}) \mid B, T \in P(\lambda_i)\}$ is a basis of the two-sided ideal $A(\Gamma_i)/A(\Gamma_{i-1})$. The isomorphisms of A-bimodule is then simpl sending $C^{\lambda_i}_{BT} + A(\Gamma_{i-1}) \mapsto v_B \otimes v_T + A^{<\lambda_i}$ where $v_B \otimes v_T \simeq C^{\lambda_i}_{BT}$. □

[1] Just a reminder.

Then, the factors at level $\lambda$of the filtration are isomorphic to direct sum of $|P(\lambda)|$ left-module V^λ. This means that, when extended to composition series, the composition factors of A are composition factors of V^λ.

With this, we can already give one of the simple modules of A.

Lemma 16. *When λ is maximal, then $V^\lambda \simeq L^\lambda$.*

Proof. We need to prove that $\mathrm{Rad}_\lambda = 0$. Suppose $v \in \mathrm{Rad}_\lambda$. We write it $v = \sum_{B\in P(\lambda)} r_B v_B$. Fix $T \in P(\lambda)$ and write $a(vT) = \sum_{B\in P(\lambda)} r_B c^\lambda_{BT} \in A$. In particular, $a(vT) \in A^{\leq\Lambda}$, and it is in $A^{<\lambda}$ if and only if $v = 0$. Since v is in the radical, we have $\langle v, w\rangle_\lambda$ for all $w \in V^\lambda$. Therefore by the definition of the bilinear form we have, for $U, S \in P(\lambda)$

$$C^\lambda_{US} a(vT) = \sum_{B\in P(\lambda)} r_B C^\lambda_{US} C^\lambda_{BT} \equiv \sum_{B\in P(\lambda)} r_B \langle v_S, v_B\rangle_\lambda C_{UT} = \langle v_S, v\rangle_\lambda C^\lambda_{UT} \overset{v\in\mathrm{Rad}_\lambda}{=} 0 \mod A^{<\lambda}$$

Then $a(vT)a \in A^{<\lambda}$ for all $a \in A^{\leq\lambda}$ and since λ is maximal, that means $a(vT)\cdot 1 \in A^{<\lambda}$ so $x = 0$. □

Theorem 17 (Graham–Lehrer)**.** *Suppose that R is a field and that Λ is finite. Then $\{L^\lambda \mid \lambda \in \Lambda^0\}$ is a complexe set of pairwise inequivalent simple modules.*

Proof. Done in class. See either [Mat99, Theorem 2.16] or [Bow25, Theorem 6.2.20] for proof. In Chris' proof, the chain of ideals comes from Lemma 15. □

This seems all dandy and fine, but sometimes it will be hard to find Λ^0. Still, we have reduced a dfficult problem of abstract algebra into a much more manageable linear algebra problem.

Next lecture (13-05-2025) we will see that there is even more to those cellular algebras, and that they will also let us state meaningful results on the, much more elusive, *indecomposable modules*.

Problem sheet 4

0. (Drill)

1. Prove that the algebra of $n\times n$ matrices is cellular with respect to the cellular datum $\Lambda = \{n\}$, $P(n) = \{1, \dots, n\}$ $C_{ij} = E_{ij}$, the elementary matrices. (it amounts to checking that $AE_{ij} = \sum_{i'} r^a_{ii'} E_{i'j}$)
2. Define $V^\lambda *$, the right A-cell module.
3. Fill the details of Lemma 7.

1. Chapter 6.3 Read Chapter 6.3 of Chris' book to get an example of a different kind of cellular algebra.

2. Chapter 6.4 Read Chapter 6.4 of Chris' book. This gives a cell structure on $\mathbb{F}\mathfrak{S}_3$. In particular, it does not suppose the field has characteristic 0.

3. Some fun with TL We call the matrices $G_\lambda = (\langle v, w\rangle_\lambda)_{v,w\in V^\lambda}$ the *Gram* matrices of the bilinear form.

Compute all the Gram matrices for $\mathrm{TL}_n(\beta)$ for $n = 2, 3, 4, 5$. Exhibit the values of β where the algebra is not semisimple.

4. Gram determinant (Difficult) Find a recursion formula for the Gram determinant of Temperley–Lieb algebra.

References

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