Problem sheet 3, 28-04-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de

0. (Drill)

1. Compute the tensor product of the matrix $\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with itself.

2. Do the computation of the Hamiltonian H_{XXZ} we saw in class to get expression for the summands E_i

$$H_{XXZ} = \frac{-1}{2}\sum_{i=1}^{n-1} \sigma_i^X \sigma_{i+1}^X + \sigma_i^Y \sigma_{i+1}^Y + \Delta \sigma_i^Z \sigma_{i+1}^Z + \delta(\sigma_i^Z - \sigma_{i+1}^Z) - \Delta(\mathrm{id}_i \otimes \mathrm{id}_{i+1})$$

   where

$$A_i = \mathrm{id}_2 \otimes \mathrm{id}_2 \otimes \cdots \otimes \underset{i}{a} \otimes \mathrm{id}_2 \otimes \cdots \otimes \mathrm{id}_2,$$

   and

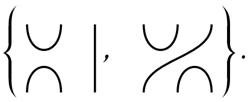
$$\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3. Verify the relations the E_i you found respect.

1. Chapter 6.1 Read Chapter 6.1 of Chris' book.

2. Modules of Temperley–Lieb Consider the Temperley–Lieb algebra $\mathsf{TL}_3(\beta)$ as a left-module on itself. Can you decompose it into irreducible components? (Block-diagonalise simultaneously the matrices of the identity, s_1 and s_2.)

3. More on one Temperley–Lieb module Consider the left $\mathsf{TL}_3(\beta)$-module $\mathsf{TL}_3(\beta)s_1$ with basis [verify]

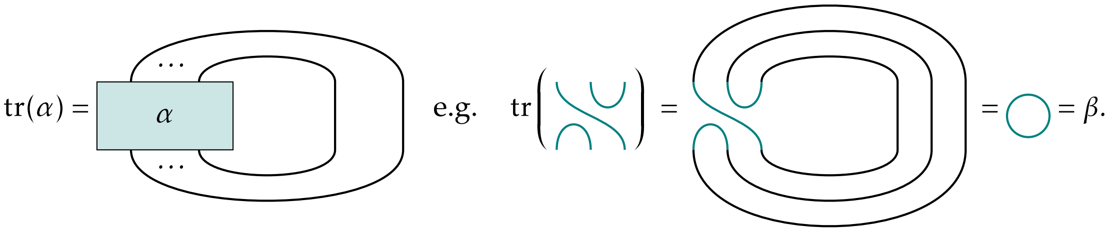


Compute the action of $\mathsf{TL}_3(\beta)$ on these module with (left) action given by (under) concatenation and evaluation. Are there value of β for which the structure of this representation changes?

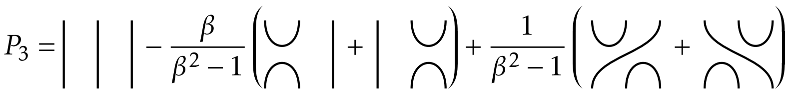
4. Preparing cellularity Write down the basis of TL_5. Give a statement on the number of through-strands acts with respect to the multiplication. Then do some numerology with respect to the number of diagrams with a given number of through strands. Compare this with the dimension of the Temperley–Lieb algebra[1].

[1] This should really tickle your (semisimple algebraically closed characteristic 0 field) representation theoretists sense. If it does not, please go contemplate Maschke's Theorem or some character—if it still is not clear, maybe check if you missed some diagrams!

5. A trace on the algebra We define a "trace" on the Temperley–Lieb algebra $\mathrm{tr} : \mathsf{TL}_n(\beta) \to \mathbb{C}$ by doing the following diagrammatic construction: given a Temperley–Lieb diagram α, embed it into a bigger space and connect the top and bottom strands with loops and compute the trace via the diagrammatic rule:



Compute the trace of the following element of $\mathsf{TL}_3(\beta)$



6. More on the pesky element Compute the actions of the two generators of $\mathsf{Tl}_3(\beta)$ on P_3.