Problem sheet 2, 14-04-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de

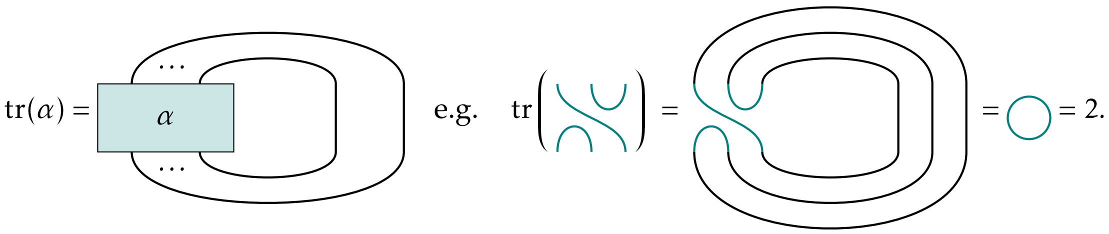
0. (Drill)

1. Write the (14) elements of $\mathsf{TL}_4(2)$ in diagram form and compute a few examples of multiplication.
2. Write the (14) walks on $\mathbb{Z}_2$ from $(0,0)$ to $(4,4)$ that do not cross the diagonal and relate them to the Temperley–Lieb diagram.
3. Choose a (big, say at least $n \geq 7$) Temperley–Lieb diagram and express it via a product of simple arcs (as we did in the proof that Ψ was surjective).

1. Catalan combinatorics Give a proof that the number of walks on $\mathbb{Z}^2$ from $(0,0)$ to (n,n) that does not cross the diagonal is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. (Hint: you might find it easier to count walks from $(0,0)$ to (n,p) that do not cross the diagonal and then specialise.)

2. Complete the proofs In the lecture, we did some of the proofs only for examples. Go back to your notes and add the necessary "$\cdots$" to make them work for all n.

3. A trace on the algebra We define a "trace" on the Temperley–Lieb algebra $\mathrm{tr} : \mathsf{TL}_n(2) \to \mathbb{C}$ by doing the following diagrammatic construction: given a Temperley–Lieb diagram α, embed it into a bigger space and connect the top and bottom strands with loops and compute the trace via the diagrammatic rule:



What is the trace of the identity? Relate this to the comment about Schur–Weyl duality in the course that stated that the Temperley–Lieb algebra $\mathsf{TL}_n(2)$ was the endomorphism algebra $\mathrm{End}_{U(\mathfrak{sl}_2)}((\mathbb{C}^2)^{\otimes n})$. Does this make sense to you?

4. Maps on the Temperley–Lieb algebras Flipping the diagram with respect to the horizontal axis gives an anti-involution (that is, $\iota^2 = \mathrm{id}$ and $\iota(ab) = \iota(b)\iota(a)$) on the (diagrammatic) Temperley–Lieb algebra. Define this anti-involution by its action on the generators.

5. Preparing the next course The diagrammatic rules we have is

$$\bigcirc = 2$$

Can we change it by

$$\bigcirc = \beta,$$

for a $\beta \in \mathbb{C}$?

We saw that we needed to have contractible loops close to 2 to be coherent with the symmetric group diagrammatic presentation. What this question asks is: "does this make sense diagrammatically" and then it begs "is there a dual structure for this new diagrammatic algebra where we have a similar Schur–Weyl duality?" [Those questions go beyond the scope of the course, hence why they get hidden here on the second page.]