Diagrammatic algebra in representation theory

Selected topics in representation theory

Lecture notes
Universität Bonn, Summer Semester 2025

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Introduction

Warning

This is a work in progress. You can let me know of typos and mistake by mail at langlois@uni-bonn.de.
The last version will be available at https://alexisl-r.github.io/teaching/S2025_Bonn_Diagrammatic/.
This version: May 19, 2025.

These notes complement the course "Diagrammatic algebra in representation theory" given at the University of Bonn during the Summer Semester 2025.

Mostly, the course follows the book "Diagrammatic algebra" of Chris Bowman [Bow25].

Exceptionally, we followed

- D. Ridout and Y. Saint-Aubin. "Standard modules, induction and the structure of the Temperley-Lieb algebra". *Adv. Theor. Math. Phys.* 18 (2014), pp. 957–1041. arXiv: 1204.4505

for dealing with Temperley–Lieb algebras and supplemented some general theory of finite-dimensional algebra from the references

- A. Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*. Vol. 15. American Mathematical Soc., 1999
- C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Vol. 356. American Mathematical Soc., 1966
- I Assem, D Simson, and A Skowroński. *Elements of Representation Theory of Associative Algebras. Volume 1. Techniques of Representation Theory*. Vol. 65. London Mathematical Society Student Texts. New York: Cambridge University Press, 1997. DOI: 10.1017/CBO9780511614309

When the lectures followed Chris' book, no note are included, but the chapters are given. Problem sheets are also available on the webpage (and included at the end as extra).

Alexis Langlois-Rémillard,
May 19, 2025

Lecture 1

`Introduction -- symmetric group -- presentation by generators and relations -- Coxeter groups -- other diagrammatic constructions`

This lecture followed roughly Chapters 1 and 2 (Sections 1.4, 2.1, 2.2, 2.4, 2.5, 2.6)

Lecture 2

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Temperley--Lieb algebra -- dimension -- Proof of the equivalence between the
diagrammatic and the generators and relations presentations.
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The lecture followed, up to a 90° shift for the diagrams, the reference was Ridout–Saint-Aubin (first sections).

D. Ridout and Y. Saint-Aubin. "Standard modules, induction and the structure of the Temperley-Lieb algebra". *Adv. Theor. Math. Phys.* 18 (2014), pp. 957–1041. arXiv: 1204.4505.

The relevant part of the book were Sections 5.1, 5.2, 6.1.

Lecture 3

`End of the equivalence proof for TL -- Motivation from physics -- generalisations of TL`

This lecture gave some physical motivation to consider Temperley–Lieb algebras and other diagrammatics.

2025-05-17: My manuscript notes are online at the page course, they will be TEXed here soon.

Lecture 4

`Refresher on finite-dimensional algebra representation theory -- Cellular algebra -- Cell modules -- Simple modules in cellular algebras`

This lecture presented cellular theory in a bit more details than the book and using a more standard definition of cellularity. I reserved the version of the book for weighted cellular algebra (Chapter 5). In particular, this allows us to treat Temperley–Lieb algebras $\mathsf{TL}_n(\beta)$ at $\beta = 0$, which I find is an interesting example.

The relevant parts of the book were Sections 6.1, 6.2, 6.3

Extra material and preliminaries of representation theory of algebra

This is a small compendium of results for finite-dimensional algebras. In general, we will not need that much so they will only be stated here without proof. References for this are the appendix of the book of Mathas [Mat99] and the classic of Curtis–Reiner [CR66].

Let A be a finite-dimensional algebra over a field $\mathbb{F}$. We will assume everything is over $\mathbb{C}$ to not trouble ourselves. Let M be a finite-dimensional A-module. We say M is a *simple* (or *irreducible*) if M is a proper and has no non-trivial proposer submodule. A *filtration* of M is a sequence of A-submodules of A

$$0 = M_0 \subset M_1 \subset M_1 \subset \ldots \subset M_k \subset M_{k+1} = M.$$

A *composition series* of M is a filtration of M where each *composition factor* M_i/M_{i-1} is a simple A-module.

Lemma 1. *Every A-module M has a composition series.*

Proof. Induction on the dimension of M. □

In particular, A viewed as an A-module on itself also has a composition series.

Lemma 2. *Suppose L is a simple A-module. Then $L \simeq A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ and L is a composition factor of the A-module A.*

One of the key results in the representation theory of finite-dimensional algebras is that those composition series are well-defined and that, even though you will be able to filter a module in many ways, its composition factors are unique up to reordering.

Theorem 3 (Jordan–Hölder)**.** *Suppose that M is an A-module and that*

$$0 = M_1 \subset \ldots \subset M_k \subset M_{k+1} = M, \qquad 0 = N_1 \subset \ldots \subset N_l \subset N_{l+1} = M$$

are two composition series of M. Then $k = l$ and for each simple module A-module L

$$|\{i \mid L \simeq M_{i+1}/M_i\}| = |\{i \mid L \simeq N_{i+1}/N_i\}|.$$

Remark 4. *The Jordan–Hölder Theorem does not hold for $\mathbb{Z}$-modules.*

In particular, the Jordan–Hölder Theorem lets us define the *composition multiplicities* $[M : L]$ of L in M as the number of composition factors of M that are isomorphic to L.

Extra material on the course

Since I departed slightly from Chris's convention for cellularity (and also used left-module to keep in line with the general notion), I will provide some course notes. Feel free to follow Chris' convention instead of mine. They are slightly less general, but most cellular algebras of relevance fall in Chris' convention. I follow here mostly the original paper of Graham and Lehrer [GL96] with some proofs taken from the book of Mathas [Mat99]. Whatch out if you are to look into the book of Mathas: he considers right-modules and his order is the reverse of the other sources (and mine in particular).

We begin with the original definition of cellular algebras due to Graham–Lehrer. Here, we put R a commutative ring with unit and our algebra are associative and unital.

Definition 5 ([GL96, Definition 1.1])**.** *A* cellular algebra *is an R-algebra together with a* cell datum $(\Lambda, P, C, *)$ *where*

- $\Lambda = (\Lambda, \leq)$ *is a poset;*
- *for each $\lambda \in \Lambda$, $P(\lambda)$ is a finite set;*
- $C : \bigsqcup_{\lambda \in \Lambda} P(\lambda) \times P(\lambda) \to A$ *is an injective map whose image is an R-basis of A. We write $C(B,T) = C^\lambda_{BT}$ for $B, T \in P(\lambda)$. This basis is called the* cellular basis *of A;*
- $* : A \to A$ *is an anti-involution,*

and the following relations are respected

$$aC^\lambda_{BT} \equiv \sum_{S \in P(\lambda)} r^a_{SB} C^\lambda_{ST} \mod A^{<\lambda} \tag{4.1}$$

$$(C^\lambda_{BT})^* = C^\lambda_{TB}, \tag{4.2}$$

where $A^{<\lambda}$ is generated by $\{C^\mu_{B'T'} \mid \mu < \lambda, B', T' \in P(\mu)\}$.

It will be useful to combine (4.1) and (4.2):

$$C^\lambda_{BT} a \equiv \sum_{U \in P(\lambda)} r^a_{BU} C^\lambda_{BU} \mod A^{<\lambda} \tag{4.3}$$

The whole point of the definition is that this gives us, on the nose, a family of modules coming from the $P(\lambda)$. We fix for the next part, a cellular algebra over a field $R (= \mathbb{C})$ with cell datum $(\Lambda, P, C, *)$.

Definition 6. *We define the (left) cell module V^λ for $\lambda \in \Lambda$ to be the free vector space with basis $\{v_B \mid B \in P(\lambda)\}$ and action given by*

$$av_B = \sum_{S \in P(\lambda)} r^a_{BB'} v_{B'}$$

where $r^a_{BB'}$ is determined by aC^λ_{BB}.

More precisely, we define V^λ_T as the R-submodule of $A^{\leq\lambda}/A^{<\lambda}$ with basis $\{C^\lambda_{BT} + A^{<\lambda} \mid B \in P(\lambda)\}$. It is a left A-module by (4.1), and furthermore it is independent of T so we can identify it with V^λ.

Observe that we can use the anti-involution to define right A-modules.

We now want to define the bilinear form. We will use a technical lemma to make sure it is well-defined.

Lemma 7. *Suppose $B, T \in P(\lambda)$. Then there exists an elements $r_{BT} \in R$ such that, for any $S, U \in P(\lambda)$*

$$C^\lambda_{ST} C^\lambda_{BU} \equiv r_{BT} C^\lambda_{SU} \mod A^{<\lambda}.$$

Proof. We simply simplify the product in two ways, first with (4.1) and then with (4.3); what will remain is simply one coefficients r_{BT}. □

This allows us to give a bilinear form $\langle -, - \rangle_\lambda$.

Definition 8. *Define a bilinear form $\langle -, - \rangle_\lambda : V^\lambda \times V^\lambda \to R$ on the basis of V^λ by*

$$\langle v_B, v_T \rangle_\lambda = r_{BT} \mod A^{<\lambda},$$

and extending linearly.

The form is symmetric and associative.

Proposition 9. *Let $\lambda \in \Lambda$, $B, T \in P(\lambda)$, $a \in A$ and $v, w \in V^\lambda$.*

1. *$\langle v, w \rangle_\lambda = \langle w, v \rangle_\lambda$.*
2. *$\langle av, w \rangle_\lambda = \langle v, a^* w \rangle_\lambda$.*
3. *$C^\lambda_{BT} v = \langle v, v_T \rangle_\lambda v_B$.*

Proof. Since everything can be done on the basis $v = v_S, w = v_U$ and extended linearly, i) follows easily by application of $*$; ii) follows from the definition of the bilinear form by choosing the parentheses: $\langle av_S, v_U \rangle_\lambda C^\lambda_{UV} \equiv aC^\lambda_{SS} C^\lambda_{UU} \equiv (C^\lambda_{SS} a^*) C^\lambda_{UU} \equiv C^\lambda_{SS}(a^* C^\lambda_{UU}) \equiv \langle v_S, v_U \rangle_\lambda C^\lambda_{SU}$. Finally, the last is precisely the definition of the bilinear form. □

Lemma 10. *Suppose $v \in V^\lambda$ and $a \in A^{\leq\mu}$. Then $\lambda < \mu \Rightarrow av = 0$.*

Proof. Apply iii) of the previous proposition. □

Definition 11. *Let $\mathrm{Rad}_\lambda = \{v \in V^\lambda \mid \langle v, w \rangle_\lambda = 0, \forall w \in V^\lambda\}$ be the radical of the bilinear form. Denote also $L^\lambda := V^\lambda / \mathrm{Rad}_\lambda$.*

We denote $\Lambda^0 = \{\lambda \in \Lambda \mid \mathrm{Rad}_\lambda \neq V^\lambda\}$.

Proposition 12. *Let R be a field*[1]*; then* Rad_λ *is the unique maximal submodule of* V^λ *and* L^λ *is simple.*

Proof. In class. $\square$

Proposition 13. *Let R be a field and let $\lambda, \mu \in \Lambda^0$. Let M be a proposer submodule of V^μ and suppose that $\sigma : V^\lambda \to V^\mu/M$ is an A-modules morphism.*

1. *If $\sigma \neq 0$ then $\mu \leq \lambda$.*
2. *If $\mu = \lambda$ then $\sigma(v) = M + r_\sigma v$ for all $v \in V^\lambda$.*

Proof. Use the strategy of the proof of Proposition 12 to use that there exists elements $v, w \in V^\lambda$ such that $\langle v, w\rangle = 1$ and then use v to generate all $v_B \in V^\lambda$. So for v_B we get $\sigma(v_B) = \sigma(a_B v) = a_B\sigma(v) + M$ and then $\lambda \prec \mu$ implies $a_B\sigma(v) = 0$ by Lemma 10. If $\lambda = \mu$, then express $a_B = \sum_{U\in P(\lambda)} r_U C_{BT}$ and thus Proposition 12 iii) implies $\sigma(v_B) = \sigma(v)a_B + M = \sum_{U\in P(\lambda)} r_U \sigma(v) C_{BU}\sigma(v) = \langle \sigma(v), y\rangle v_B$ for $y = \sum_{U\in P(\lambda)} r_U v_U$, and $r_\sigma = \langle \sigma(v), y\rangle_\lambda$. $\square$

Corollary 14. *The L^λ are pairwise non-isomorphic.*

Proof. In class. $\square$

We now prove a technical lemma giving a filtration of the cellular algebra.

Lemma 15. *Suppose that Λ is finite with $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_k = \Lambda$ is a maximal chains of ideal of Λ. Then we can find a total ordering $\lambda_1, \ldots, \lambda_k$ such that $\Gamma_i = \{\lambda_1, \ldots, \lambda_i\}$. Then*

$$0 = A(\Gamma_0) \hookrightarrow A(\Gamma_1) \hookrightarrow \ldots \hookrightarrow A(\Gamma_k) = A$$

is a filtration of A with composition factors $A(\Gamma_i)/A(\Gamma_{i-1}) \simeq V^{\lambda_i *} \otimes_R V^{\lambda_i}$.

Proof. Since the chain of ideal is maximal, that means $\Gamma_i \setminus \Gamma_{i-1} = \{\mu\}$ for a certain μ. So we can find a total ordering. As a consequence, $A^{<\lambda_i} \subset A(\Gamma_{i-1})$ and the basis $\{C^{\lambda_i}_{BT} + A(\Gamma_{i+1}) \mid B, T \in P(\lambda_i)\}$ is a basis of the two-sided ideal $A(\Gamma_i)/A(\Gamma_{i-1})$. The isomorphisms of A-bimodule is then simple sending $C^{\lambda_i}_{BT} + A(\Gamma_{i-1}) \mapsto v_B \otimes v_T + A^{<\lambda_i}$ where $v_B \otimes v_T \simeq C^{\lambda_i}_{BT}$. $\square$

Then, the factors at level $\lambda$of the filtration are isomorphic to direct sum of $|P(\lambda)|$ left-module V^λ. This means that, when extended to composition series, the composition factors of A are composition factors of V^λ.

With this, we can already give one of the simple modules of A.

Lemma 16. *When λ is maximal, then $V^\lambda \simeq L^\lambda$.*

Proof. We need to prove that $\mathrm{Rad}_\lambda = 0$. Suppose $v \in \mathrm{Rad}_\lambda$. We write it $v = \sum_{B\in P(\lambda)} r_B v_B$. Fix $T \in P(\lambda)$ and write $a(vT) = \sum_{B\in P(\lambda)} r_B c^\lambda_{BT} \in A$. In particular, $a(vT) \in A^{\leq\Lambda}$, and it is in $A^{<\lambda}$ if and only if $v = 0$. Since v is in the radical, we have $\langle v, w\rangle_\lambda$ for all $w \in V^\lambda$. Therefore by the definition of the bilinear form we have, for $U, S \in P(\lambda)$

$$C^\lambda_{US} a(vT) = \sum_{B\in P(\lambda)} r_B C^\lambda_{US} C^\lambda_{BT} \equiv \sum_{B\in P(\lambda)} r_B \langle v_S, v_B\rangle_\lambda C_{UT} = \langle v_S, v\rangle_\lambda C^\lambda_{UT} \overset{v\in\mathrm{Rad}_\lambda}{=} 0 \mod A^{<\lambda}$$

Then $a(vT)a \in A^{<\lambda}$ for all $a \in A^{\leq\lambda}$ and since λ is maximal, that means $a(vT)\cdot 1 \in A^{<\lambda}$ so $x = 0$. $\square$

[1] Just a reminder.

Theorem 17 (Graham–Lehrer)**.** *Suppose that R is a field and that Λ is finite. Then $\{L^\lambda \mid \lambda \in \Lambda^0\}$ is a complete set of pairwise inequivalent simple modules.*

Proof. Done in class. See either [Mat99, Theorem 2.16] or [Bow25, Theorem 6.2.20] for proof. In Chris' proof, the chain of ideals comes from Lemma 15. □

This seems all dandy and fine, but sometimes it will be hard to find Λ^0. Still, we have reduced a difficult problem of abstract algebra into a much more manageable linear algebra problem.

Next lecture (13-05-2025) we will see that there is even more to those cellular algebras, and that they will also let us state meaningful results on the, much more elusive, *indecomposable modules*.

Lecture 5

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Cellular algebra -- indecomposable modules -- composition series -- Jordan--Hölder
Theorem -- decomposition number -- Krull--Schmidt Theorem -- idempotents --
radical
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Let $\mathcal{A}$ be a finite-dimensional algebra. When seen as a left-module on itself, it has a composition series, that is, a filtration where the factor are simple:

$$0 = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_k \subset \mathcal{A}_{k+1} = \mathcal{A}, \tag{5.1}$$

wehere $\mathcal{A}_i/\mathcal{A}_{i-1}$ is simple.

Lemma 18. *Any simple $\mathcal{A}$-module M appears a as a composition factor of $\mathcal{A}$.*

By the Jordan–Hölder Theorem 3, we can speak of the Jordan–Hölder composition series of a module.

Definition 19. *The decomposition number of a simple module L inside a module M is the number of time L appears as a composition factor inside the composition series of M. We denote it* $[M : L]$.

In cellular algebra, we define the decomposition matrix as

$$\mathrm{D} = ([V^\lambda : L^\mu])_{\lambda\in\Lambda,\mu\in\Lambda_0}. \tag{5.2}$$

As a consequence of Graham Lehrer Theorem 17, we know how D looks like.

Corollary 20. *The matrix* D *is unitriangular.*

In fact, we have even more, as cellular algebra also gives us information on the more elusive indecomposable modules.

Definition 21. *A module M is indecomposable if it does not decompose into the direct sum of two non-trivial submodule.*

In particular, simple modules are indecomposable, but the converse is not always true.

From a course of representation theory of finite groups over $\mathbb{C}$, one might get this impression, but on algebra, it is easy to find example where it does not work.

We will be interested in a class of special indecomposable modules.

Theorem 22 (Krull–Schmidt)**.** *Suppose M is an A-module and that*

$$M_1 \oplus \ldots \oplus M_k = M = N_1 \oplus \ldots \oplus N_l$$

are two decomposition of M into a direct sum of indecomposable modules. Then $k = l$ and we can rearrange the N_i via a permutation σ such that $N_{\sigma i} \simeq M_i$.

We will call the indecomposable module P_i appearing in the Krull–Schmidt decomposition of $\mathcal{A}$ viewed as a module on itself, the principal indecomposable.

The radical $\mathrm{Rad}(\mathcal{A})$ of a finite-dimensional $\mathbb{F}$-algebra $\mathcal{A}$ was defined as the sum of all nilpotent ideals (so ideal I for which there exists an $n \in \mathbb{N}$ such that $I^n = 0$). Then we define the radical of a submodule $M \subset \mathcal{A}$ as $\mathrm{Rad}(M) := \mathrm{Rad}(\mathcal{A}) \cap M$.

There is a one-to-one correspondence between principal indecomposable module and simple module given by sending $P \mapsto P/\operatorname{Rad} P$.

In cellular algebra, we can then speak of the matrix given by the decomposition number $[P^\lambda : L^\mu]$ for $\lambda, \mu \in \Lambda_0$. By denoting P^λ the principal indecomposable whose head is isomorphic to L^λ $(P^\lambda/\operatorname{Rad} P^\lambda \simeq L^\lambda)$. We denote $\mathrm{C} = ([P^\lambda : L^\mu])_{\lambda,\mu\in\Lambda_0}$

A wonderful result of cellular theory is that we can access these decomposition number via the decomposition matrix.

Theorem 23 (Graham–Lehrer)**.** *Let A be a cellular algebra over a field with Λ finite. Then*

$$\mathrm{C} = \mathrm{D}^t\mathrm{D}.$$

Lecture 6

`ℤ-gradings -- weighted cellular algebras -- ℤ--graded (weighted) cellular algebras -- The Idempotent Trick -- The Grading Trick -- binary Schur algebra`

This lectures covered material from the books. Relevant sections are Sections 5.3, 5.6, 5.7, 6.3 (gradings and binary Schur algebras); Sections 6.6, 6.7, 6.9 (weighted cellular algebras and $\mathbb{Z}$–graded cellular algebra)

Lecture 7

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