Lecture 3

27·04·2025

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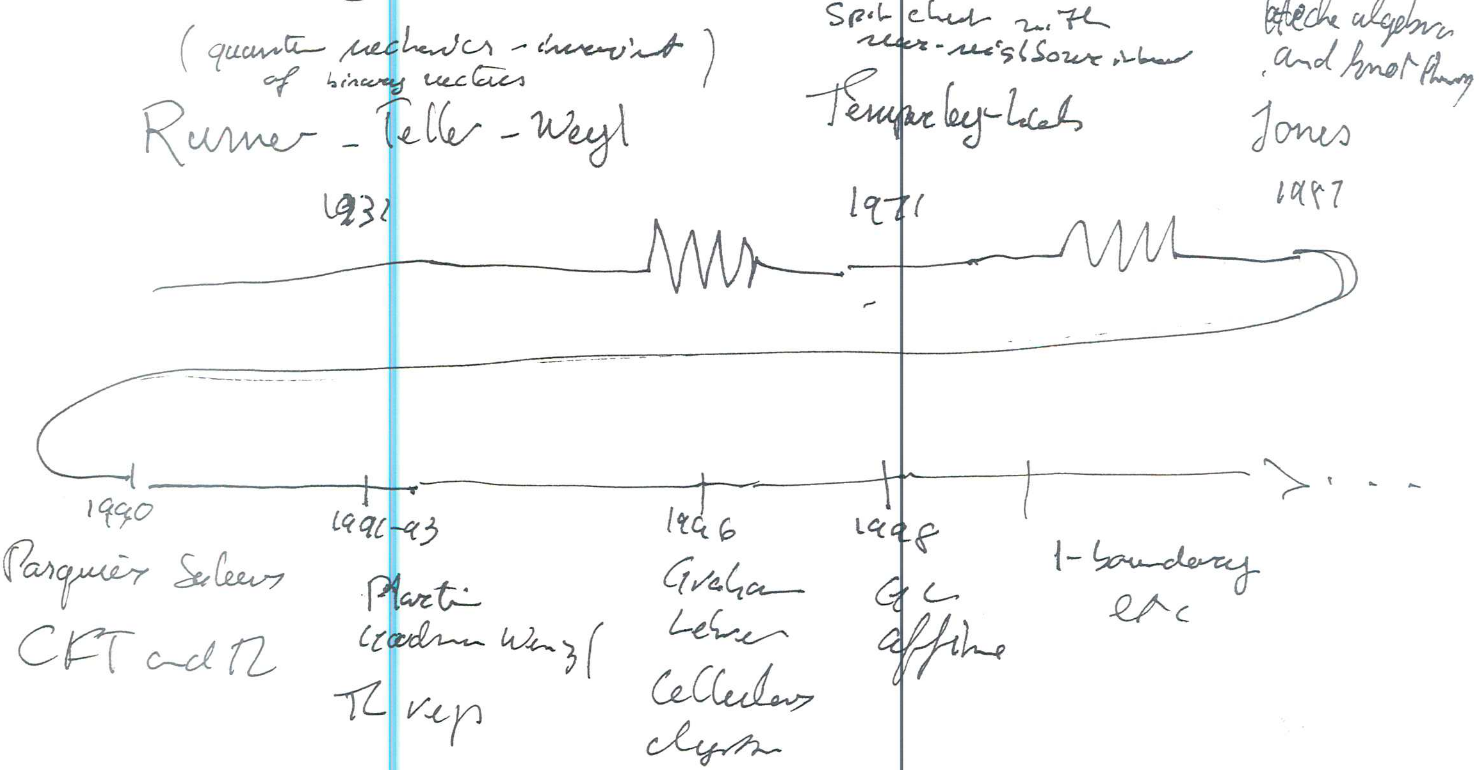
Diagrammatic algebra in representation theory — Bonn

Some physical motivations for diagrammatic algebra.

Bit of history.



A bit of physics

Given a set of particles, we define an Hamiltonian that will define the evolution of the system.

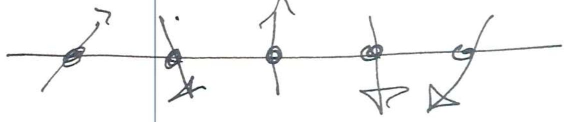
Take particle σ_i with $i \in I$ indexing their position. ~~the~~ takes value in the potential space of admissible states, say $\{\sigma\}$.

The Hamiltonian goes: $H : \{a\} \longrightarrow \{a\}$

Example: Spin chain XXZ.

We have n particles interacting only with their closest neighbours.

the state of a particle is its spin: a combination of $\uparrow$ and $\downarrow$ (so $\mathbb{C}^2$)



The Hamiltonian describing it has interaction between nearest neighbours: spin at position i interacts with $i-1$ and $i+1$

$$H_{XXZ} = \frac{-1}{2} \sum_{i=1}^{n-1} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + \delta(\sigma_i^z - \sigma_{i+1}^z) - \Delta(id_i \otimes id_{i+1})$$

Where we denote via the shorthand

$$X_i = \underset{1}{Id_2} \otimes \underset{2}{Id_2} \otimes \cdots \otimes \underset{i}{X} \otimes \underset{i+1}{Id_2} \otimes \cdots \otimes \underset{n}{Id_2}$$

and $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(Pauli matrices)

Let's compute the 4x4 matrixes

$$\sigma_i^x \sigma_{i+1}^x = \left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \hline \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{array}\right) \qquad \sigma_i^y \sigma_{i+1}^y = \left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \\ \hline \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{array}\right)$$

$$\sigma_i^z \sigma_i^z = \left(\begin{array}{cc|cc} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \hline \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{array}\right) \qquad id_i\, id_{i+1} = \left(\begin{array}{cc|cc} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \hline \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{array}\right)$$

Now we add everything in the Hamiltonian and we get:

$$= \begin{pmatrix} \ddots & & & \\ & -\delta+\Delta & -1 & \\ & -1 & \delta+\Delta & \\ & & & \ddots \end{pmatrix} \qquad \text{Write} \quad \Delta = \frac{q+q^{-1}}{2}, \quad \delta = \frac{-q+q^{-1}}{2}$$

$$E_i := id_2 \otimes \cdots \otimes id_2 \otimes \begin{pmatrix} \ddots & & & \\ & q & -1 & \\ & -1 & q^{-1} & \\ & & & \ddots \end{pmatrix} \otimes id_2 \otimes \cdots \otimes id_2.$$

$i \quad i+1$

Now if we compute the properties of these linear transformations, we get

$E_i^2 = (q+q^{-1})E_i \qquad 1 \leq i < n-1$

$E_i E_{i+1} E_i = E_i \qquad 1 \leq i \leq n-2$

$E_i E_{i-1} E_i = E_i \qquad 2 \leq i \leq n-1$

$E_i E_j = E_j E_i \qquad |i-j| > 1$

Putting $q=1 \rightsquigarrow$ The Temperley-Lieb algebra appears again! $(TL_n(2))$

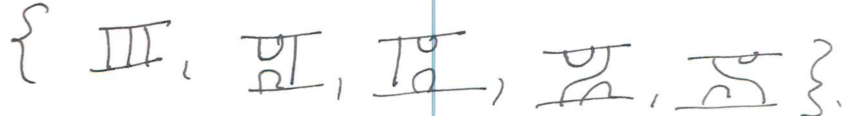
Now, we can already see some ways to generalise this, but first, there is a problem: is it still well-defined?

Interlude: is $TL_n(q+q^{-1})$ well-defined for $q \in \mathbb{C}^\times$

Interestingly: the physical model depends on the value of q. Does the algebra $TL_n(q+q^{-1})$ too?

Ex. Let's have a look at TL_3.

We look at it as a module on itself

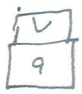
Bases: 

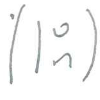
$\beta = q+q^{-1}$

$\rho($  $) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & \beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \beta \end{pmatrix}$

$\rho: TL_3 \longrightarrow End_{TL_3}$

$a \longmapsto a\cdot$

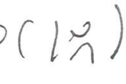
$a \cdot v =$ 

$\rho($  $) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \beta & 0 & 1 \\ 0 & 1 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

~~Let's take the right submodule M~~

~~with basis~~

Take  as a basis (left submodule)

$\rho_2($  $) = \begin{pmatrix} \beta & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \rho($  $) = \begin{pmatrix} 0 & 0 \\ 1 & \beta \end{pmatrix}$

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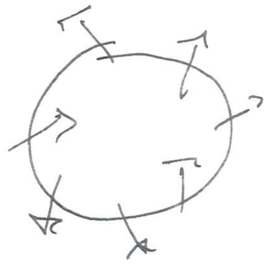
Other models

1. If we add periodic condition for the particles of the spin chain we have still an Hamiltonian



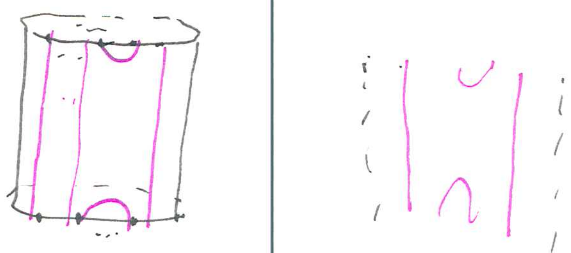
$$H_{XXZ} : \sum_{i=1}^{N} E_i$$

$$E_i = \sigma_i^- \sigma_{i+1}^+ + \sigma_i^+ \sigma_{i+1}^- + (q+q^{-1})\sigma_i^+\sigma_i^- \sigma_{i+1}^+ \sigma_{i+1}^- - q^{\pm 1}\sigma_i^+\sigma_i^- + q^{\mp 1}\sigma_{i+1}^+\sigma_{i+1}^-$$

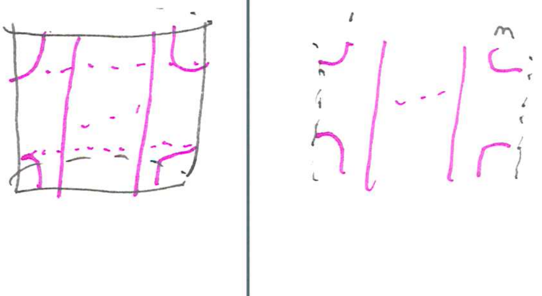
$$E_N = \zeta^2 \sigma_N^- \sigma_1^+ + \zeta^{-2} \sigma_N^+ \sigma_1^- + (q+q^{-1})\sigma_N^+\sigma_N^- \sigma_1^+\sigma_1^- - q^{\pm 1}\sigma_N^+\sigma_N^- - q^{\mp}\sigma_1^+\sigma_1^-$$

This corresponds to a periodic version of the Temperley-Lieb algebra with diagrams on the cylinder

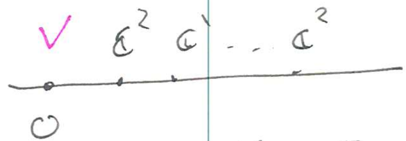
$E_i \rightsquigarrow$



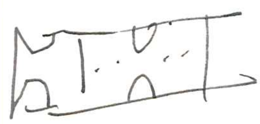
$E_N \rightsquigarrow$



2. If we add a boundary condition on the spin chain (this amounts to changing one of the boundary to another state.



this amounts diagrammatically to adding a boundary operators (with certain [illegible]

E_i  E_0: 

with relate like $E_0^2 = \delta E_0$

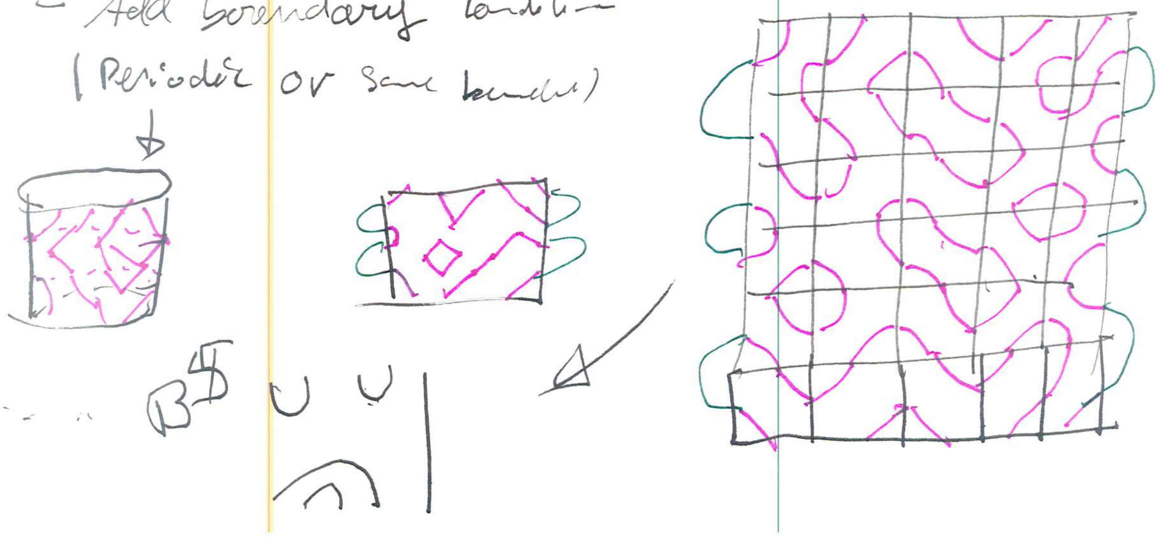
and $E_1 E_0 E_1 = \delta_2 E_1$

$E_0 E_1 E_0 = \delta_3 E_0$.

3. Loop model and percolation.

Put two tiling  with certain weight on a lattice

- Add boundary condition (Periodic or same boundary)



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Similarly pick $\{|\cup\cap, \cup\cap|\}$

$$\rho_3(\cup\cap|) = \begin{pmatrix}0&0\\1&\beta\end{pmatrix}, \quad \rho_3(|\cup\cap) = \begin{pmatrix}0&0\\\beta&1\end{pmatrix}.$$

$\rho_2 \simeq \rho_3$

Furthermore, take

$$v = ||| - \frac{\delta}{\delta^2-1}\left(\cup\cap| + |\cup\cap\right) + \frac{1}{\delta^2-1}\, \cap\!\!\!\smile + \smile\!\!\!\cap$$

Trust me that

$$\rho_4(id) = 1, \quad \rho_4(\cup\cap|) = 0 \quad \rho_4(|\cup\cap) = 0.$$

The general rep can be written as

$$\rho = \rho_4 \oplus \rho_2 \oplus \rho_3.$$

Furthermore, in general ρ_4, ρ_2 are the simple modules and we have

$$\dim TL_3 = 5 = \dim\rho_4^2 + \dim\rho_2^2 = 1+4.$$

__BUT__ when $\beta = 1$, it is not the case anymore that ρ_2 is simple!

(it's not completely reducible, but it has an invariant sub module

We will see more next course