Queens of the chessboards

A collection of mathematical chess puzzles

Alexis Langlois-Rémillard

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Preface

A chess problem is genuine mathematics, but it is in some way "trivial" mathematics. However, ingenious and intricate, however original and surprising the moves, there is something essential lacking. Chess problems are unimportant. The best mathematics is serious as well as beautiful – "important" if you like, but the word is very ambiguous, and "serious" expresses what I mean much better.

G. H. Hardy

This work is an attempt to collect some of the material I accumulated over the years on chess and mathematics. After some years of being a mathematician, and many more a chess player, the examples lying at the intersection of those two interested gradually accumulated. It was always in the back of my mind to combine the two, so I kept a mental note each time I saw something interesting combining them. What follows is an organic document tenting to share what I found.

I am certainly not the first to assemble chess-inspired mathematics. I do direct you to the books of Watkins [9] and the collection of problem assembled by Petković [6] for an interesting read. A lot of the references and ideas I got for the queens problem come from the nice survey by Bell and Brett [2]. An older and lighter source for the n-Queens problem is the piece of Rivin, Vardi and Zimmermann in the American Mathematical Monthly [8].

Chapter 1 is a translation of an outreach article [4] written for the Québec magazine *Accromath*. It gives an overview of the Eight queen problem and presents how to attack it with a computer. Chapter 2 is a translation of outreach article [5] written with Charles Senécal for *Accromath* following up on the one presented in Chapter 1. It goes over generalisations of the Eight queens and queen domination problems happening when the geometry of the chessboard is changed: precisely what happens if we consider a chessboard on a torus, as Pólya did [7] or try to dominate polyominoes, as recently Alpert and Roldán did [1]. Chapter 3 then presents a collection of problems around those themes. Part of them were put with the *Accromath* articles and a few were made for an activity of the mathematical student association PRIME at UGent in 2021. Finally, Chapter 4 contains further reading and plans for future activities I would like to try.

A number of people was bothered along the way with questions related to this project, and will undoubtedly still be; it is then only justice that I take a moment to thank them for their involvement.

Thanks to Charles Senécal for participating in this project with me. Thanks to Jean-Philippe Chassé for reading an earlier draft and diplomatically telling me "it was certainly not the best thing you made", hopefully the current version will get more approval. Thanks to Érika Roldán for encouraging me to translate the work on queen problem and explaining their results with Hannah Alpert. Thanks to Christoph Müßig for sharing his thoughts on the chess art gallery polyomino game. Thanks to Steven Van Overberghe and his friends at PRIME to have suffered my Dutch long enough to try the puzzles and share their solutions. Thanks to Hadewijch De Clercq for help with the Dutch version of the problems. Finally, many thanks to Toon Baeyens for spending some time working his C magic and making vague koffie pauze ideeën true.

Alexis Langlois-Rémillard, last updated: September 7, 2022

Warning

It is a work in progress, I welcome any comments on it you may have. Sending an email at alexislangloisremillard@gmail.com should do the trick. Since I love snail mail, I offer postcard reward for typos or contributions: just make sure you are looking at the latest version (and write me where to send them)! The latest version can be found on my website at: `alexisl-r.github.io/popularization/echecs/`

Chapter 1

Eight queens and a chessboard

By Alexis Langlois-Rémillard[1]

One chessboard, eight queens and a constraint, that is all the needed component of the Eight queens puzzle. Despite this simplicity, it keeps being studied after more than 170 years. We follow the traces of one of the greatest mathematician to unveil its secrets.

1.1 A very special puzzle

Chess as we know it has been played for centuries, and variants of the game, for thousands of years. Next to the game itself, enthousiasts and amateurs have also created many a puzzles around using the rules of the game. If some of those puzzles were created by teachers as routine exercises to improve and practice, others were build by aesthetes that have developed chess problem creation into an art, crafting rightfully-called compositions under strict rules.

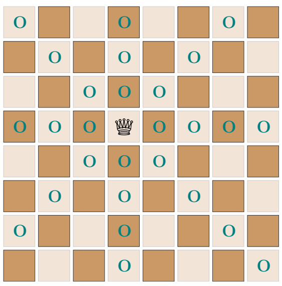


Figure 1.1: Movement of the queen

Outside this spectrum, another type of puzzle was discussed next to the black and white terrain: amusement using chess pieces and their geometrical movements with no concern for the game. Most famous amongst those is the knight tour: one has to make a knight visit each square of the chessboard only once. Often, those problems thought as entertainment hid deep mathematical ideas in the simplicity of their questions, often drawing professionals and amateurs alike to try their luck at the study. One of the best example of the impact of such puzzle needs only queens and sparked a century and a half quest to unravel its secrets. A queen *threaten* another if she can reach the square the other occupies (see Figure 1.2).

The *Eight queen puzzle* was first published in 1848 by Max Bezzel (1824–1871) in the German periodic *Schachzeitung* (chess journal). It was then showcased many times in other chess revues and other newspapers.

Problem 1.1.1. *How many ways are there to place eight queens on a chessboard without having them threaten each others?*

[1]This chapter is a translation, with small modifications, of Alexis Langlois-Rémillard. "Huit dames et un échiquier". In: *Accromath* 17.1 (2022), pp. 8–13. ISSN: 1911-0189. URL: https://accromath.uqam.ca/2022/02/huit-dames-et-un-echiquier/.

To begin, a queen moves on a chessboard alongside rows, columns and diagonals (see Figure 2.3). If it is (relatively) easy to find one solution to the problem, how can we be sure to find them all? Before keeping reading, now would be a good time to take a pause, find a (possibly virtual) chessboard, gather some queens and try!

At the intersection of chess and maths

Where does this association between the worlds of chess and mathematics in popular culture? Looking at the history, we can easily understand why! In the sixteen world champions since 1862, we find two professional mathematicians: Emanuel Lasker (champion from 1894 to 1921) and Machgielis Euwe (champion from 1935 to 1937), and one engineer and computer scientist: Mikhael Boitvinnik (champion from 1948 to 1957, 1958 to 1960 and 1961 to 1963). Still today, many top players studied mathematics, for example the grandmasters John Nunn, Thomas Ernst, Jonathan Speelman, Jonathan Mestel and Kasten Müller, to name a few, all have a PhD in mathematics. At the very top, the 2021 blitz and 2009 junior world champion Maxime Vachier-Lagrave has taken the time to finish his bachelor in mathematics before becoming a full-time professional.

Chess, with its rules pertaining to the movement of the pieces in its space of 64 squares, inspired throughout the years many problems mathematically rich. In the eighteenth century, Leonhard Euler (1707–1783) studied with a new field, graph theory, the knight's tour problem, a famous puzzle where it is asked to visit all squares of the chessboard once and only once using a knight. Later, the German mathematician Ernst Zemerlo (1871–1953) used chess as an example to establish the bases of the field of game theory by defining mathematically the concept of winning position in his contribution to the International Congress of Mathematician in 1913.

In the middle of the twentieth century, the game inspired mathematicians to apply their knowledge to create programs that could play chess. Alan Turing (1912–1954) created such a program in 1948 with his friend the statistician Davig G. Champernowne (1912–2000). Computer then could not run the program as it was too complex, so the computations had to be done manually! The game was for a long time a lab to test methods in artificial intelligence. For example, when the mathematician Barbara Liskov (1939–) developed heuristics of optimal search for her PhD thesis in 1968, she applied them to a program specifically created to play chess endgames to showcase their power.

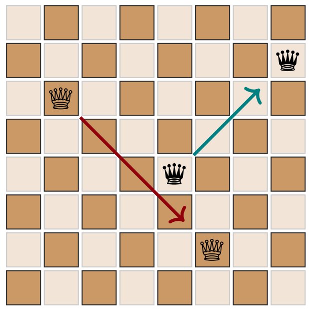


Figure 1.2: The queens in red are threatening themselves along a SE diagonal, ant those in blue, along a NE diagonal.

1.2 The solution of the Prince of mathematics: Gauss

There are many ways to solve this problem, as much maybe as great persons trying to solve it. We will follow the lead of the astronomer Heinrich Christian Schumacher (1780–1850). Schumacher was passionate about chess, interested in mathematics and kept all his life many correspondences. One man with whom he exchanged often was the great German mathematician Carl Friedrich Gauss (1777–1855).

Gauss wrote to him about the problem in September 1850 after seeing an article of Franz Nauck in the Leipzig gazette *Illustrirten Zeitung*. Nauck claimed to have solved the problem and gave 60 solutions. Intrigued, Gauss had spent a morning on it and found more solutions and asked his friend if he had seen the puzzle.

Schumacher was quite enthusiast and the two exchanged a few letters about it. Meanwhile, Nauck corrected his mistake and correctly gave the 92 solutions. Gauss did not verify that they were all of them, but he explained a method to do so to his friend, letting him finish the computations stating: "With those methodic trial-and-errors, one would not find it difficult to find the solutions if one would be ready to spend an hour of two[2]". The story does not tell if Schumacher solved the challenge before his death, but the method of Gauss is certainly still worth considering.

It goes like this, step by step

1. The movement of a queen is the union of four lines: a vertical, a horizontal and two diagonal, one North-East of slope 1 and one South-East of slope -1.

2. One and only one queen is required on each column and one and only one queen is required on each row. So all the vertical and horizontal lines must be distinct.

3. From this we note the position the queens by a list of 8 numbers

$$(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8)$$

   where Q_i gives the row on which the queen of the ith column is, counting from bottom to top. This make sure that there is only one queen per column. When all columns are distinct, all the Q_i are distinct and we call such a list a *permutation*. All solutions must be permutations, but not all permutations are solutions: diagonals matter.

4. Two queens in positions (j, D_j) and (k, D_k) are on the same NE diagonal if the two coordinate pairs lie on the same line of equation $y = x + b$, so if $D_j - j = D_k - k$. For example on Figure 1.2, the two black queens are in position $(5, 4)$ and $(8, 7)$, and they are on the line $y = x - 1$ as $7 - 8 = -1 = 4 - 5$.

5. For the SE diagonals, we follow the same line of reasoning, but for lines of equation $y = -x + b$. For example, on Figure 1.2 again, the white queens on position $(2, 6)$ and $(6, 2)$ are on the line $y = -x + 8$.

6. The condition for a permutation to be a solution is thus: all sums $D_k + k$ must be distinct for $1 \leq k \leq 8$ and all differences $D_k - k$ also.

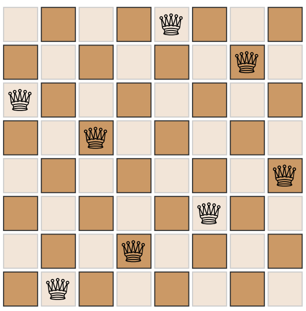


Figure 1.3: A solution to the 8 queens problem.

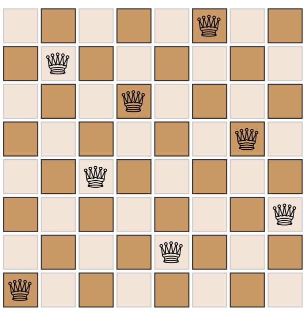


Figure 1.4: Not a solution to the 8 queens problem.

One example and one non-example: Figure 1.3 corresponds to to the permutation $(6, 1, 5, 2, 8, 3, 7, 4)$. We quickly compute all differences and all sums to get that all numbers are different in the difference (A) and in the sum (B).

	6	1	5	2	8	3	7	4		6	1	5	2	8	3	7	4
-	1	2	3	4	5	6	7	8	+	1	2	3	4	5	6	7	8
A	5	-1	2	-2	3	-3	0	-4	B	7	3	8	6	13	9	14	12

However, if we consider the the position of Figure 1.4, which corresponds to the permutation $(1, 7, 4, 6, 2, 8, 5, 3)$, we see that it is not a solution since there are two "2" in the difference (C) and two "7" in the sum (D).

[2]"Schwer ist es übrigens nicht, durch ein methodisches Tatonnireu sich diese Gewissheit zu verschaffen, wenn man 1 oder en Paar Stunden daran wenden will."

	1	7	4	6	2	8	5	3		1	7	4	6	2	8	5	3
-	1	2	3	4	5	6	7	8	+	1	2	3	4	5	6	7	8
C	0	5	1	2	-3	2	-2	-5	D	2	9	7	10	7	14	12	11

After long verifications, one can obtain 92 unique solutions to the problem. These solutions can then be grouped in 12 families. Eleven of eight solutions are given by one solution of the family and the images under rotation of 90°, 180° and 270°, and under the four reflections with respect to the horizontal, vertical and diagonal axes. A last family of four solutions is given by one solution of the family and its image under the rotation of 90° and under the two reflections with respect to the vertical and diagonal NE axes.

Following blindly Gauss's method would yield

$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40\,320$$

permutations to check. The great German mathematician was maybe a bit optimist when he said it would be possible to verify everything in one or two hours! But we might expect somebody of the stature of Gauss to have some tricks under his sleeves.

1.3 A step backward, two forward

Let us assume you would want to find all solutions to the Eight queen problem. You would undoubtedly become quite good at those computations, an would quickly find ways to save some work. For example, it is clear that any permutation starting by $(1, 2, x, x, x, x, x, x)$ is not a solution since the two first queens attack themselves. By discarding all such permutations, you just saved $6! = 720$ positions to check, not bad!

This is good, but how can we be certain to find as many such heuristics as possible, and more importantly, be sure that all the solutions will be found? This is a commonly studied question in informatics. One way to ensure this is called *backtracking*. The Eight queens puzzle is the classical example to illustrate those type of algorithms.

In the context of the puzzle, the idea of the algorithm rests upon one observation: for a permutation to be a solution on the $n \times n$ chessboard, the first k queens must be a solution on the $k \times n$ chessboard. Practically, this means that we add the queens one by one, and verify at each step that no queen gets threatened when we do so. If adding a queen creates a problem, we change the last queen(s) until we reach a solution, hence the name. This way, we never encounter two permutations containing the same problem.

1.4 And for bigger?

What happens if we try to generalise the problem for bigger chessboards? We get a generalisation of the problem. Mathematicians loves generalising problems, and this one is no exception. When Nauck got the solutions of the Eight queen puzzle, he also proposed to look into generalisation to bigger chessboards. The French mathematician Lionnet in 1869 asked this question to the students of the École normale supérieure and Polytechnique. It reads as follows.

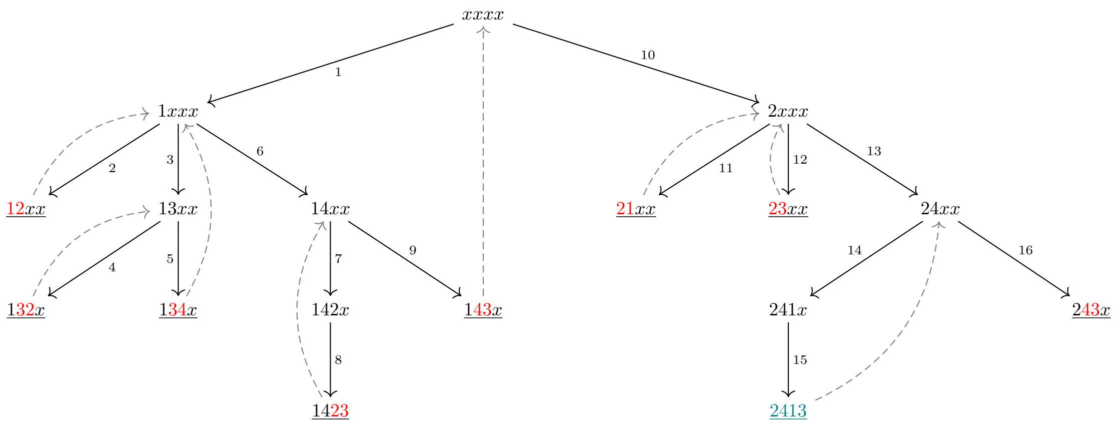


Figure 1.5: The backtracking algorithm on a 4×4 chessboard

Problem 1.4.1. *How many ways are there to place n non-threatening queens on a $n \times n$ chessboard?*

The first serious attempt recorded to tackle the problem was due to Emil Pauls, a pharmacist interested and chess and mathematics. We will see later his general solution. However, let's remark that the solution we proposed for the Eight queens problem is suitable to tackle the general problem. To paraphrase Gauss, someone with a bit of knowledge of programming would only need one or two hours to write a backtracking algorithm able to generate all solutions for a $n \times n$ chessboard.

There is only one small problem. Even if the backtracking algorithm we described is very efficient, it still grows very fast. And indeed, despite all the advances in computing power, we do not know the exact number of solutions to the n queens problem when $n > 28$. The computations for $n = 27$ took slightly more than one year with state-of-the-art massively parallel supercomputers in 2016.

We could continue to search for better algorithms, or study the problem deeper, but we will never be able to find a closed formula for all n. Indeed, Hsiang, Hsu and Shieh proved in 2004 that it is not possible [3].

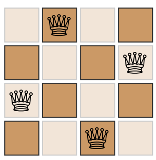


Figure 1.6: Pauls solution for 4.

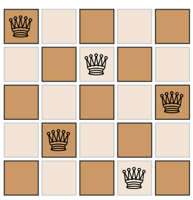


Figure 1.7: Pauls solution for $n = 5$.

In view of this result, most of the research in informatics on the n queens problem have been concentrated on finding one solution, instead of enumerating them all. Many sophisticated algorithms were originally tested on this task. The interest of such problems is to inspire methods which can then be applied elsewhere: the result is less important than the path taken. And in fact, finding one solution to the n queens problem does not require a fancy algorithm. Emil Pauls gave one construction for $n > 3$ already in 1873. We present his solution since it is quite simple and elegant. It depends on the remainder of the division of n by 6.

- $n = 6k$ or $n = 6k + 4$. The permutation with first half even numbers, and second odd numbers, both in increasing order, so

  $$(2, 4, \ldots, n, 1, 3, \ldots, n - 1)$$

  is a solution. (Figure 1.6 is an example for $n = 4$.)

- $n = 6k + 1$ or $n = 6k + 5$. Place one queen in the top left corner and place the solution for the remaining $n-1 \times n-1$ chessboard presented below. (Figure 1.7 is an example for $n = 5$.

- $n = 6k + 2$. It is the most complex case, let's follow it on Figure 1.8. First, place the left- and rightmost queens at $(1, 4)$ and $(n, n-3)$ (in green). Then, place the middle at $(n/2-1, n)$, $(n/2, 2)$, $(n/2+1, n-1)$ and $(n/2+2, 1)$ (in blue). In the left part, place then queens at each $(i, n-2(i-1))$ for $2 \leq i \leq n/2-2$ (in red). And in the right part, queens are placed at $(j, 2n-2j+1)$ for $n/2+3 \leq j \leq n-1$ (in pink).
- $n = 6k + 3$. It suffices to place the solution for $n - 1 = 6k + 2$ previously shown in the first $n-1 \times n-1$ chessboard and to add a queen in the top right corner (Figure 1.9).

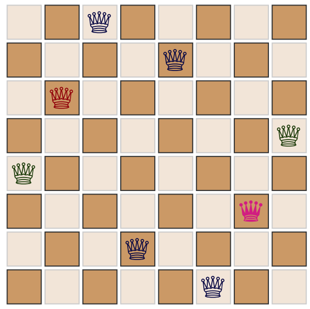


Figure 1.8: Pauls solution for $n = 8$.

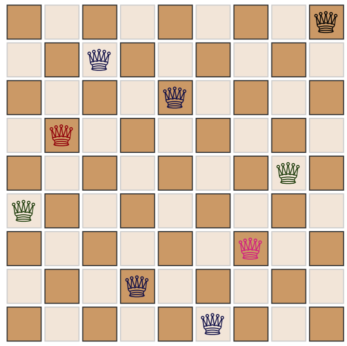


Figure 1.9: Pauls solution for $n = 9$.

Algorithmic complexity

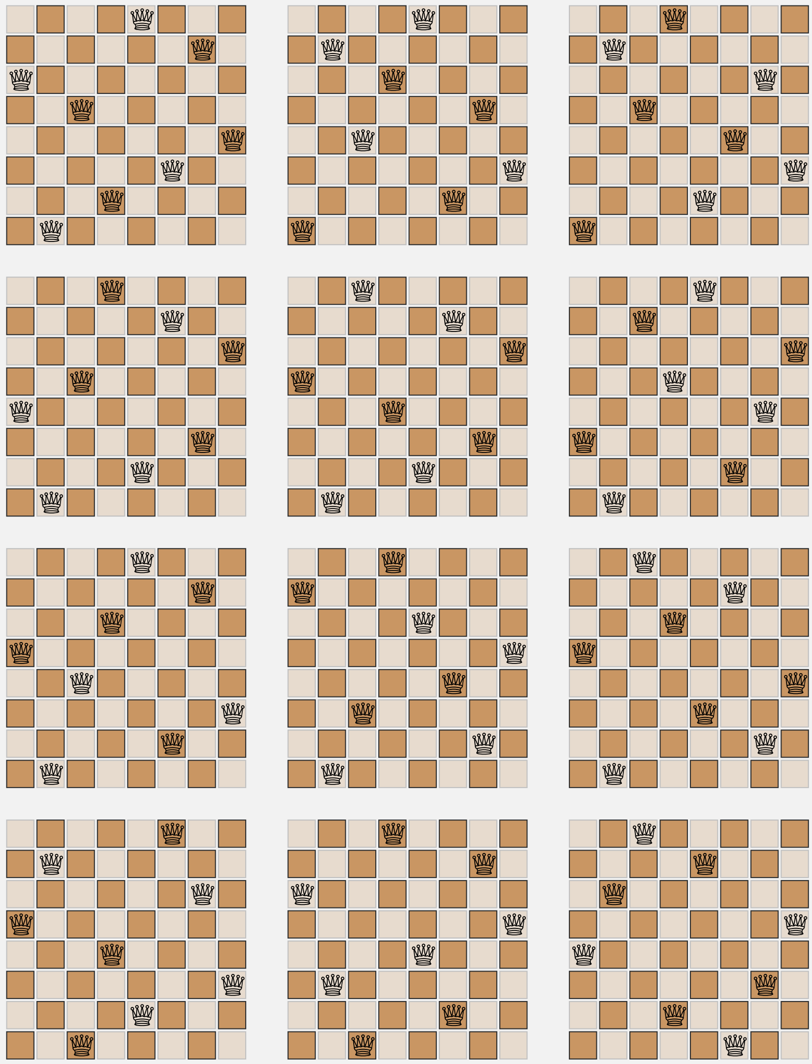
A computer has a great computation power, but it is not unlimited. Informatics has developed tools to define the complexity of algorithms. At its base, the more steps an algorithm needs to complete a task, the more complex it is. Algorithms are divided in classes according to the type of growth their complexity experiments as the size of the task increases.
The n queens problem contains many examples of complexity classes. Ingenious algorithms have been created to find a solution in polynomial time. Finding all solutions with the backtracking algorithm requires an exponential time. The brute-force permutation method of Gauss needs a factorial time, but it is still magnitude better than the naive brute-force method where we choose n squares to place queens in the n^2 of the chess board. For this one, the time is bigger-than-factorial as the number of position to verify is given by the binomial coefficient $\binom{n^2}{n} = \frac{n^2!}{n! \times (n^2-n)!}$. If it is possible to solve the Eight queens problem this way with a modern desktop, for the 20 queens problem, it would require to verify more than 2.78×10^{33} positions, which would require more than a human lifetime even with all the supercomputers of the Earth!

n	Brute	Gauss	Backtracking	One solution
1	1	1	1	1
2	6	2	1	2
3	84	6	4	6
4	1820	24	9	∼10
5	53130	120	35	∼20
8	4426165368	40320	∼1096	∼50
10	1.73×10^{14}	3628800	∼18000	∼350
14	8.71×10^{20}	87178291200	∼13679276	∼2000
n	$\binom{n^2}{n}$	$n!$	a^n	$n \log(n) - n^3$

There is still a lot to be done around this problem. We only considered here placing the maximal number of queen on a chessboard. A closely-related problem asks for the minimal amount of queens necessary to cover all squares on the chessboard. Other generalisations of the problems include checking for rectangular chessboards, or even more foreign boards or paving. In the next chapter of this work, we will present two of these generalisations.

All solutions for the Eight queens problem

The first eleven are in family of 12 solutions by rotations and reflection and the last is in a family of 4 by rotation of 90° and reflections along the horizontal and diagonal NE axes.



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Chapter 2

Queens on queer chessboards

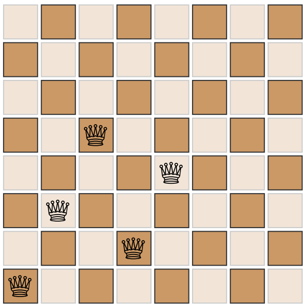

Figure 2.1: Domination of 5 queens

By Alexis Langlois-Rémillard and Charles Senécal[1]

Chess, with its simple rules governing the movement of pieces in its closed space, inspired many interesting puzzles. Some of them have surprising links with various domains of mathematics. In the dialogue between chess and mathematics, it was sometime mathematics that modified the rule of the game to let interesting problems hatch.

2.1 Generalising classical problems

One of the most celebrated mathematical-chess problem is the *Eight queen problem* covered in Chapter 1. This puzzle asks to place the maximal amount of non-threatening queens on a chessboard. A similar problem, the *Queen domination problem* is to ask the minimal amount of queens needed to cover all square of the chessboard, so that each square is either occupied by a queen or guarded by a queen.

Mathematicians who studied those two problems have long generalised them by increasing the size, and adding and removing constraints. Some were straightforward: studying a $n \times n$ chessboard instead of the classical one for example, but some modified deeply the rules.

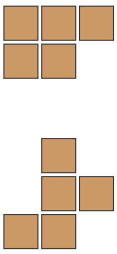

Figure 2.2: Two polyominoes

In the following, we will consider two such attempts. The first one was introduced by the Hungarian mathematician Georg Pólya in 1918 and considers what happens to the n queens problem if the chessboard is placed on a torus [7]. The second was studied a century latter by Hannah Alpert and Érika Roldán and consider the domination problem on *polyominoes* [1].

So, the two problems will be the following.

Problem 2.1.1. *For which n is there a solution for the n queens problem on the torus.*

Problem 2.1.2. *What is the minimal number of queens necessary to dominate a polyomino of N tiles.*

[1]This chapter is a translation of Alexis Langlois-Rémillard and Charles Senécal. "Des dames sur d'étranges échiquiers". In: *Accromath* (2022). to appear, 6p.

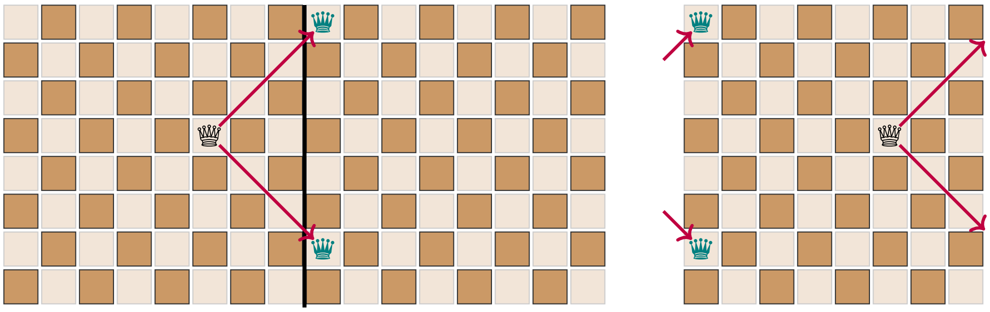

Figure 2.4: Movement of the queen, decomposed.

2.2 Pólya's generalisation with a modular detour

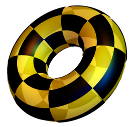

Figure 2.3: A toroidal chessboard

Let us consider the toroidal chessboard. How do the pieces move on such a donut-shaped board? We do not need to have such a strange board to think about the movement of the pieces: a simple chessboard is enough, we only need to add some special rules once we reach the border.

To transform a normal chessboard in a toroidal one, we construct a "modular" chessboard first. This special chessboard consists in many chessboards put next to each others. When they move, pieces simply continue their paths on the boards. When the movement ends, we than put the piece at the same square it reached on the initial board.

We denote the position of a queen by (c, r) for c the column, and r the row. An example of the movement described is present in Figure 2.4. There, the queen at $(6, 5)$ can reach the squares $(9, 8)$ and $(9, 2)$ on the second chessboard. This means she can reach the two squares $(1, 8)$ and $(1, 2)$ on the initial chessboard.

2.3 Modular arithmetic

Maybe the last section description did not convince you. Luckily, there is a mathematical way to describe such thing: *modular arithmetic.* In the modular world, equalities are replaced by congruences linked to a certain number n. Two numbers a, b are said to be congruent if they share the same remainder by the division by n, so if there exists an integer k such that $a = k \times n + b$. We denote congruences by $a = b \bmod n$.

The clock is a typical example of modular arithmetic for $n = 24$ (or $n = 12$). Sleeping 8 hours after going to bed at 23:00 means waking up at 7:00, not 31:00! We just did here the congruence $31 = 7 \bmod 24$.

The "modular" chessboard we introduced previously is an example of modular arithmetic for $n = 8$. All added chessboards are coming from vertical and horizontal translations of 8 units. The operation "going back to the initial chessboard" is precisely taking the modulo of each coordinate (c, r).

2.4 Pólya's problem

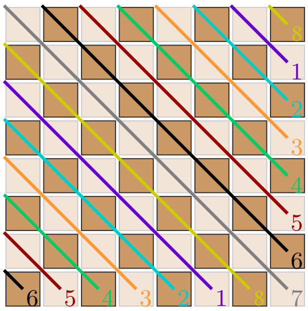

Figure 2.5: The 8 SE diagonals

1,8,7	2,8,6	3,8,5	4,8,4	5,8,3	6,8,2	7,8,1	8,8,8
1,7,8	2,7,7	3,7,6	4,7,5	5,7,4	6,7,3	7,7,2	8,7,1
1,6,1	2,6,8	3,6,7	4,6,6	5,6,5	6,6,4	7,6,3	8,6,2
1,5,2	2,5,1	3,5,8	4,5,7	5,5,6	6,5,5	7,5,4	8,5,3
1,4,3	2,4,2	3,4,1	4,4,8	5,4,7	6,4,6	7,4,5	8,4,4
1,3,4	2,3,3	3,3,2	4,3,1	5,3,8	6,3,7	7,3,6	8,3,5
1,2,5	2,2,4	3,2,3	4,2,2	5,2,1	6,2,8	7,2,7	8,2,6
1,1,6	2,1,5	3,1,4	4,1,3	5,1,2	6,1,1	7,1,8	8,1,7

Figure 2.6: The numerotation of the chessboard

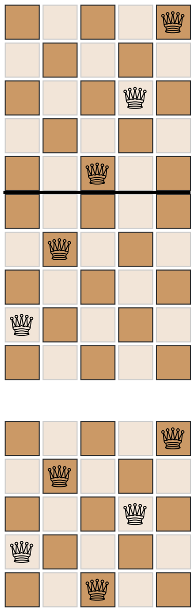

Figure 2.7: Solution of the toroidal five queens problem

The biggest difference of the toroidal n queen problem and the classical one rests in the diagonal. For a classical 8×8 chessboard, there are 15 NE diagonals; on a torus, only 8 remains (see Figure 2.5).

We know from Chapter 1 that all classical $n \times n$ chessboards have a solution for $n \geq 4$. Pólya was asking himself if putting the chessboard on a torus would change that, and if so, how? Before giving what he found, we study the problem for a 8×8 chessboard.

Let us index the SE diagonals as in Figure 2.5 and write, for each square of the chessboard, its column c, its row r and its SE diagonal d. Each of these numbers has to be between 1 and 8. The choice of numbering for SE diagonals was not random: it is made so in order that 8 divides the sum $c + r + d$ for each square, see Figure 2.6.

Suppose the set $\{(c_i, r_i, d_i) \mid 1 \leq i \leq 8\}$ is a solution of the toroidal eight queens problem. Then, all queens are on different rows, different columns and different diagonals. Therefore if we sum any of those numbers c_i, r_i or d_i, we simply sum the number from 1 to 8. Adding the three we obtain:

$$\sum_{i=1}^{8}(c_i + r_i + d_i) = \sum_{i=1}^{8} c_i + \sum_{i=1}^{8} r_i + \sum_{i=1}^{8} d_i = 3\sum_{i=1}^{8} = 3 \times 36 = 108.$$

But we have a problem: 108 is not divisible by 8. The thing is, all $c_i + r_i + d_i$ are divisible by 8, thus our supposition cannot be true: there thus are no solution to the toroidal eight queens problem.

The constraint added by Pólya is non-trivial. Is it too restrictive? If we verify the small chessboard, we quickly find that it is not: the 5×5 chessboard has a toroidal solution. It is obtained by placing a queen following the knight jump, as shown in Figure 2.7

What does the 5×5 chessboard have that the 8×8 does not? Pólya answered this question by an elegant criterium to determinate if a chessboard has toroidal solution.

Theorem 2.4.1 (Pólya, 1918). *Let $n \geq 4$. A solution to the toroidal n queens exist if and only if n and 6 are relatively prime, that is if 2 and 3 do not divide n.*

Proof

Consider a list $(r_1, \dots, r_n)$ of integer in $\{1, \dots, n\}$. The number r_i then corresponds to the position of the queen presents at the column c_i. This list is a solution of the toroidal n queens problem if and only if:

i. $(r_1, \dots, r_n)$ is a permutation of the numbers of 1 to n: each of them appears once only;

ii. $((r_1 + 1) \bmod n, \dots, (r_n + n) \bmod n)$ is a permutation, with $0 = n \bmod n$;

iii. $((r_1 - 1) \bmod n, \dots, (r_n - n) \bmod n)$ is a permutation, with $0 = n \bmod n$.

Those three conditions ensures that no two queens share the same row, the same column (i), the same diagonal SE (ii) or the same diagonal NE (iii). To prove the theorem, we must first probe that if the condition is satisfied, then there is a solution, and then that if a solution exist, then 2 and 3 do not divide n.

Suppose that 2 and 3 do not divide n. We prove that $(r_1, \dots, r_n)$ given by $r_k = 2k \bmod n$ is a solution. For the 5×5 chessboard, it is precisely the solution of Figure 2.7. We check that all three points are verified.

The point i is verified as 2 does not divide n. That means that 2 has an inverse modulo n since the $\gcd(n, 2) = 1$. (For example, $2^{-1} = 3 \bmod 5$ as $3 \times 2 = 6 = 1 \bmod 5$.) Hence, the list $2k \bmod n$ is a permutation. If $2k = 2p \bmod n$, then we can multiply both side by 2^{-1} to return to $k = p \bmod n$. The list $r_k = 2k \bmod n$ is thus a permutation.

The point ii is verified as the sum in question is simply the list $(3, 6, \dots, 3n\} \bmod n$. As 3 does not divide n, it also has an inverse modulo n and so the list is a permutation by the same argument as above.

The point iii requires us simply to remark that $r_k - k = 2k - k = k \bmod n$, which is the permutation $(1, 2, \dots, n)$.

We now show the other direction of the implication. Suppose that there is a solution. It respects then all three points i, ii and iii. We add all the elements $r_i - i$ from the third permutation. Since it is a permutation, it is the same as summing over the number from 1 to n, hence we obtain the famous formula from summing consecutive numbers:

$$\sum_{j=1}^{n} (r_j - j) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \bmod n.$$

However, we can also sum differently, and as the point i tells us $(r_1, \dots, r_n)$ is a permutation, we obtain:

$$\sum_{j=1}^{n} (r_j - j) = \sum_{j=1}^{n} r_j - \sum_{j=1}^{n} j = 0 \bmod n.$$

We combine the two equalities and obtain $\frac{n(n+1)}{2} = 0 \bmod n$. Thus, n divides $\frac{n(n+1)}{2}$. If n was even, then $n = 2^s r$ for a certain odd integer r. Then

$$\frac{n(n+1)}{2} = 2^{s-1}(2^s r + 1).$$

But this number cannot be divided by n as $2^s r + 1$ is odd and n contains s factor 2. By contradiction, n must then be odd, and so 2 does not divide n.

We now show that 3 does not divide n. To do so, we sum the square of he two lists given in points ii and iii. Both of them are permutation, so we are only summing the square of the numbers from 1 to n. By using the formula for the sum of consecutive squares we obtain

$$\sum_{j=1}^{n} (r_j - j)^2 + \sum_{j=1}^{n} (r_j + j)^2 = 2 \sum_{k=1}^{n} k^2 = 2 \frac{n(n+1)(2n+1)}{6} \bmod n.$$

If we instead develop the two squares, we obtain, as $(r_1, \ldots, r_n)$ is a permutation by point i,

$$\sum_{j=1}^{n}(r_j - j)^2 + \sum_{j=1}^{n}(r_j + j)^2 = \sum_{j=1}^{n}(r_j^2 - 2r_j + j^2) + (r_j^2 + 2r_j + j^2)$$
$$= 4\sum_{k=1}^{n} k^2 = 4\frac{n(n+1)(2n+1)}{6} \mod n.$$

We thus obtain the following congruence:

$$2\frac{n(n+1)(2n+1)}{3} = \frac{n(n+1)(2n+1)}{3} \mod n \quad \text{or} \quad \frac{n}{3} = \frac{2n}{3} \mod n.$$

The last equation is possible only if 3 does not divide n. If it did, then $n = 3^s r$ for r non-divisible by 3. However, the equation demands that $n/3 = 3^{s-1}r$ be divisible by n, a contradiction.
Therefore, if a solution exists for the toroidal n queens problem, 2 and 3 do not divide n and we proved Pólya's theorem. □

2.5 Polyominoes and domination

Let us now turn to a second generalisation of the chessboard: polyominoes. We mean by polyomino a connected set of tiles. Hence, we consider any set of tiling, not just a $n \times n$ square, as long as they are connected. By this, we mean that we can travel to any tile in the polyomino by crossing their edges. Figure 2.8 gives examples of polyominoes and non-polyominoes.

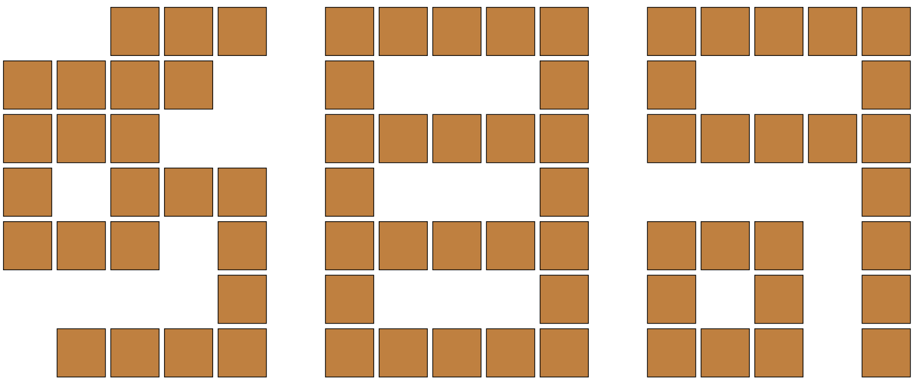

Figure 2.8: Two polyominoes of 23 and 26 tiles, and one non-polyomino (which can be seen as one polyomino of 8 tiles, and one of 16)

The problem we consider for those new strange chessboards is the domination problem. Given a polyomino of N tiles, how many queens do we need to guard all the tiles. If a given set of queens placed on the tiles of the polyomino guard all the tiles, we say that this set "dominates" the polyomino.

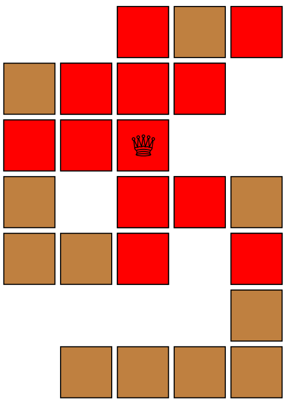

Figure 2.9: A queen and the tiles she guards.

Remark that we lost a constraint in contrast with the preceding problem: the queens can threaten themselves. Hence, a trivial solution to the polyomino domination problem is to place a queen on each tile! To investigate more interesting configurations, we will instead ask for the minimal number of queens necessary to guard a polyomino. The answer will obviously depends of the geometry of the polyomino in question. For example, the domination of a 9 tiles polyomino can require one, two or three queens as shown in Figure 2.10

A criterion was published by the mathematicians Hannah Alpert and Érika Roldán in 2021 [1]. It gives a upper bound to the minimal amount of queens necessary to guard a polyomino of N tiles.

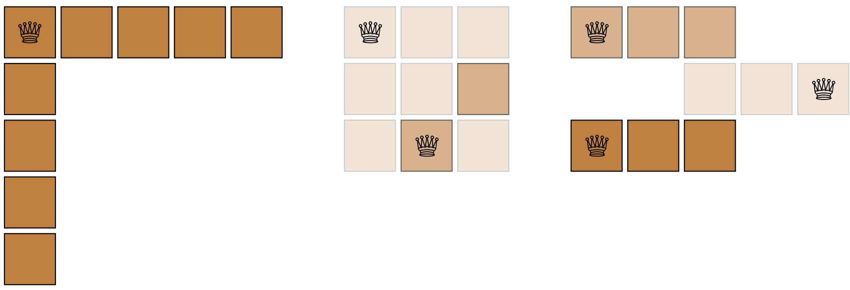

Figure 2.10: Polyominoes of 9 tiles who need 1, 2 or 3 queen(s) for their domination.

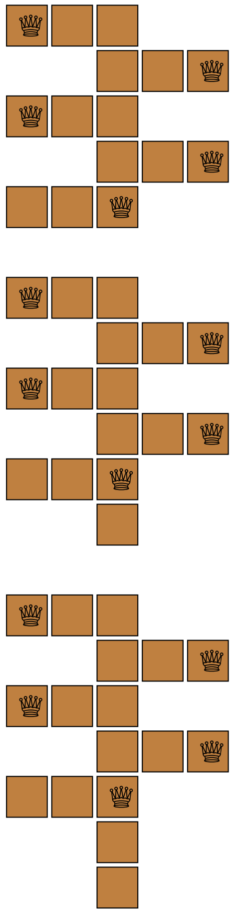

Figure 2.11: Polyominoes needing the maximal amount of queens for $N = 15, 16, 17$.

Theorem 2.5.1 (Alpert–Roldán, 2021). *The number of queens that is sufficient and sometime necessary to guard a polyomino of* $N \geq 3$ *tiles is* $\lfloor N/3 \rfloor$.

We will present a version of their proof, leaving out one subtlety as an exercise.

Proof

There are two parts to prove: first that we can cover any polyomino of N tiles with $\lfloor N/3 \rfloor$ queens, and second that some polyominoes require $\lfloor N/3 \rfloor$ queens.

We begin with the latter. We construct such polyominoes by putting on top of each other stacks of lines of 3 tiles sharing one straight stem of $\lfloor N/3 \rfloor$ tiles. The remaining tiles are then added at the beginning of the stem. It is best represented by an example in Figure 2.11.

We now show that any polyomino of N tiles can be guarded by $\lfloor N/3 \rfloor$ queens. To do so, we first define the distance in between two tiles by the length of the shortest path in between the two, with the rule that we can only travel from one tile to the ones on its left, right, top or bottom. Furthermore, we can only travel on the polyomino.

With this notion of distance we can remark that a queen guard all tiles at distance at most 2 of her. She guards her own tiles, so the tile at distance 0. All tiles at distance 1 are only a movement away, and the tiles at distance 2 are either in a straight line from her, or at one queen movement away in diagonal. Figure 2.12 shows one example.

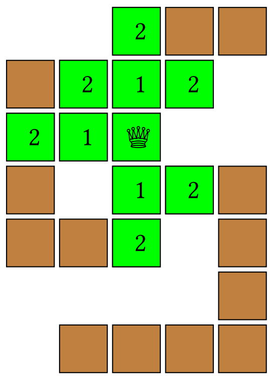

Figure 2.12: A queen and the tiles at distance at most 2 from her.

Given a polyomino of N tiles, we choose one tile that we name the *root* of the polyomino. If possible, we choose the root to be a tile that only touch another tile, if this is not possible, we choose any tile[2]. Now, we note on all the tiles of the polyomino their distance to the root tile. We will colour the polyomino with three colours. The colour we use will depend of the distance modulo 3: one for the tiles at distance $0 \bmod 3$; one for those at distance $1 \bmod 3$ and a last one for those at $2 \bmod 3$. All tile have a colour, and the least represented colour appears at most $\lfloor N/3 \rfloor$ times. We then place a queen on all tile of the least represented colour. Then, from any tile, taking the path to the root will ensure that we cross a queen in at most two steps. By the preceding remark, the queen guard the tile. All tiles of the polyomino are then guarded by at least one queen. An example of the construction is given in Figure 2.13. □

With a bit of creativity in mathematics, we often explores interesting and unexpected results by modifying a given problem. One only needs to ask: "what if?"

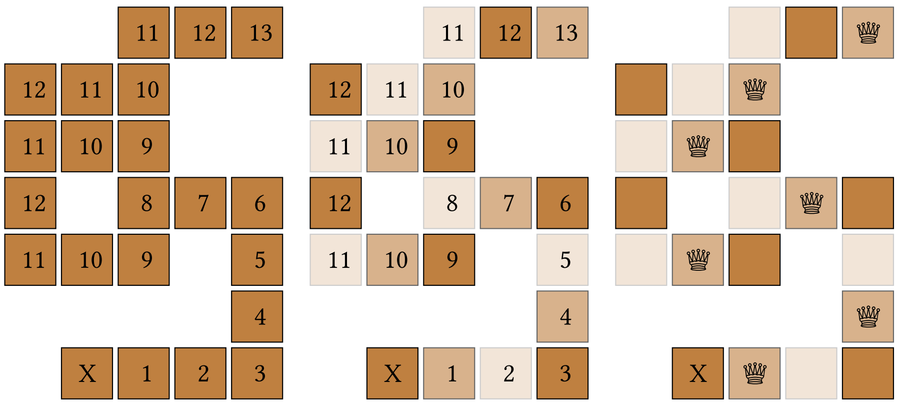

Figure 2.13: The three steps of the domination by 7 queens of a 22 tiles polyomino.

Bibliography

[1] Hannah Alpert and Érika Roldán. "Art Gallery Problem with Rook and Queen Vision". In: *Graphs and Combinatorics* 37.2 (2021), pp. 621–642. DOI: 10.1007/s00373-020-02272-8 (cit. on pp. 5, 15, 20).

[2] Jordan Bell and Brett Stevens. "A survey of known results and research areas for n-queens". In: *Discrete Mathematics* 309.1 (2009), pp. 1–31. DOI: 10.1016/j.disc.2007.12.043 (cit. on p. 5).

[5] Alexis Langlois-Rémillard and Charles Senécal. "Des dames sur d'étranges échiquiers". In: *Accromath* (2022). to appear, 6p. (Cit. on pp. 5, 15).

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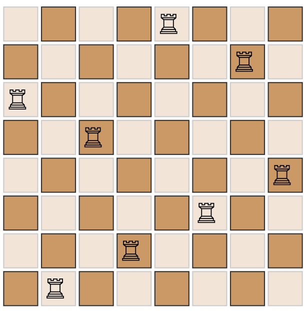
[2]This choice is where the subtlety lies, we cover it in more details in Exercise 3.2.3.

Chapter 3

Exercises

13	25	7	19	1
17	4	11	23	10
21	4	20	2	14
5	12	24	6	18
9	16	3	15	22

Figure 3.1: Magic square of magical constant 65



$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.1 Queens on the chessboard

Exercise 3.1.1. *In the middle of the eighteenth century, the African mathematician Muhammad ibn Muhammad studied magic squares*[1]*: $n \times n$ tables filled with numbers from 1 to n^2 such that the sum of the lines, or the diagonals or the column is always equal to a magical constant. In one of his manuscript, he gave a construction of magical squares using knight moves. Can you give a link with the five queens problem?*

Exercise 3.1.2. *Prove that there are $n!$ permutations of n elements and give the correspondence of permutation and the n rooks problem.*

Exercise 3.1.3. *Take a solution of the n rooks problem. Replace each rook by a 1, and each empty square by a 0 to obtain a $n \times n$ matrix. What can you say about those matrices?*

3.2 Queens on queer chessboards

Exercise 3.2.1. *Prove the beginning of Pólya's proof.*
Consider a list $(r_1, \dots, r_n)$ of integer in $\{1, \dots, n\}$. The number r_i then corresponds to the position of the queen presents at the column c_i. This list is a solution of the toroidal n queens problem if and only if:

i. $(r_1, \dots, r_n)$ *is a permutation of the numbers of 1 to n: each of them appears once only;*

ii. $((r_1 + 1) \bmod n, \dots, (r_n + n) \bmod n)$ *is a permutation, with $0 = n \bmod n$;*

iii. $((r_1 - 1) \bmod n, \dots, (r_n - n) \bmod n)$ *is a permutation, with $0 = n \bmod n$.*

Exercise 3.2.2. *Give the equivalent result for the polyomino domination problem if we change queens for rooks. (Hint: what would be the tiling used?)*

[1] Those objects were believed to display magical properties. As such, it was a tradition to always place mistakes in the book where they were explained so that only the initiates could access their magical properties.

Exercise 3.2.3. *What would happen if we tried the same construction as in the chapter but choosing the root as in Figure 3.2.*

Exercise 3.2.4. *What happens if one of the colours is empty in the proof of Alpert–Roldán's theorem?*

Exercise 3.2.5. *The theorem proved by Alpert and Roldán was in fact for generalised polyominoes, called polycubes, in d dimensions. How would the situation changes for tridimensional polyocubes?*

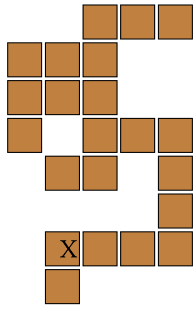


Figure 3.2: Other choice of root

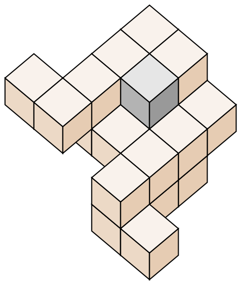


Figure 3.3: A 3D polycube

3.3 Other exercises

Chapter 4

For more

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