DIAGRAMMATICS FOR LAX AND FROBENIUS MONOIDAL FUNCTORS AND WEAK MORPHISM CLASSIFIERS

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ABSTRACT. The theory of 2-monads entails that, for a strict monoidal category C, there is a strict monoidal category $\mathsf{L}(\mathsf{C})$ such that strict monoidal functors from $\mathsf{L}(\mathsf{C})$ are precisely the lax monoidal functors from C. We give an elementary, diagrammatic, construction of $\mathsf{L}(\mathsf{C})$ and of its variants for oplax and Frobenius lax functors. The diagrams used are analogous to the diagrammatics for lax monoidal functors studied by McCurdy.

1. Introduction

The different notions of weak homomorphisms of monoidal categories, provided by lax, oplax and Frobenius monoidal functors, have in recent years been the subject of increased interest outside of pure category theory, as exemplified by applications in the theory of Hopf monads (see [16, 4]), of tensor categories and their Drinfeld centres [9, 8, 7, 12, 11], quantisation and infinitesimal braidings [22], virtual tangles [3], topological theories generalising TQFTs [10], and condensation in modular fusion categories [20].

A diagrammatic calculus for such functors has been developed independently on a number of occasions, including [17, 22, 20], building on ideas of [5] and [19]. These accounts vary in the extent of formality and completeness, with [17] being particularly extensive. The diagrams for Frobenius monoidal functors naturally extend the familiar diagrammatics for Frobenius algebras, which, in turn, resemble a "flat" variant of two-dimensional TQFTs; see [15].

In this note, we show that, given a (strict) monoidal category C, the diagrams mentioned above can be used to define a new monoidal category $\mathsf{L}(\mathsf{C})$ (as well as an oplax and a Frobenius lax variant thereof), such that lax (resp. oplax, Frobenius lax) monoidal functors out of C are equivalently the strict monoidal functors out of $\mathsf{L}(\mathsf{C})$ (resp. $\mathsf{opL}(\mathsf{C})$, $\mathsf{F}(\mathsf{C})$).

This not only immediately verifies the soundness of said diagrammatic calculus, but also provides an explicit, and quite elegant, answer to the question of the existence of a category satisfying the universal property of $\mathsf{L}(\mathsf{C})$ described above. In the language of category theory, we verify the existence of a *lax morphism classifier* (see [14, Section 2.4]) for the 2-monad on **Cat** defining monoidal categories. Precisely this question was raised by John Baez in [1], and while the affirmative answer to it can be deduced from the general theory of 2-monads (see [2, Theorem 3.13], [13, Theorem 2.4]), our direct approach is both more elementary and it gives a more complete answer in this particular case, since we do not merely prove the existence of $\mathsf{L}(\mathsf{C})$, but also describe it explicitly.

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2. Construction of classifiers for (Frobenius) lax monoidal functors

Let $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ be a functor of strict monoidal categories. A *lax monoidal structure* on $\mathcal{F}$ consists of a family of morphisms $m_{X,Y} : \mathcal{F}(X) \otimes_{\mathsf{D}} \mathcal{F}(Y) \to \mathcal{F}(X \otimes_{\mathsf{C}} Y)$ and $u : \mathbb{1}_{\mathsf{D}} \to \mathcal{F}(\mathbb{1}_{\mathsf{C}})$ natural in $X, Y \in \mathsf{C}$ and such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{F}(X)\otimes\mathcal{F}(Y)\otimes\mathcal{F}(Z) & \xrightarrow{m_{X,Y}\otimes\mathcal{F}(Z)} & \mathcal{F}(X\otimes Y)\otimes\mathcal{F}(Z) \\ {\scriptstyle \mathcal{F}(X)\otimes m_{Y,Z}}\downarrow & & \downarrow{\scriptstyle m_{X\otimes Y,Z}} \\ \mathcal{F}(X)\otimes\mathcal{F}(Y\otimes Z) & \xrightarrow[m_{X,Y\otimes Z}]{} & \mathcal{F}(X\otimes Y\otimes Z) \end{array} \quad ; \tag{2.1}$$

$$\begin{array}{ccc} \mathbb{1}_{\mathsf{D}}\otimes\mathcal{F}(X) & \xrightarrow{u\otimes\mathcal{F}(X)} & \mathcal{F}(\mathbb{1}_{\mathsf{C}})\otimes\mathcal{F}(X) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle m_{\mathbb{1}_{\mathsf{C}},X}} \\ \mathcal{F}(X) & \xrightarrow{=} & \mathcal{F}(\mathbb{1}_{\mathsf{C}}\otimes X) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}(X)\otimes\mathbb{1}_{\mathsf{D}} & \xrightarrow{\mathcal{F}(X)\otimes u} & \mathcal{F}(X)\otimes\mathcal{F}(\mathbb{1}_{\mathsf{C}}) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle m_{X,\mathbb{1}_{\mathsf{C}}}} \\ \mathcal{F}(X) & \xrightarrow{=} & \mathcal{F}(X\otimes\mathbb{1}_{\mathsf{C}}) \end{array} \quad . \tag{2.2}$$

An *oplax monoidal structure* on $\mathcal{F}$ consists of morphisms $c_{X,Y} : \mathcal{F}(X \otimes_{\mathsf{C}} Y) \to \mathcal{F}(X) \otimes_{\mathsf{D}} \mathcal{F}(Y)$ and $e : \mathcal{F}(\mathbb{1}_{\mathsf{C}}) \to \mathbb{1}_{\mathsf{D}}$ endowing $\mathcal{F}^{\mathrm{op}}$ with the structure of a lax monoidal functor. A *Frobenius monoidal structure* on $\mathcal{F}$ consists of a lax monoidal structure $(m_{X,Y}, u)$ on $\mathcal{F}$ and an oplax monoidal structure $(c_{X,Y}, e)$ on $\mathcal{F}$, such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{F}(X)\otimes\mathcal{F}(Y\otimes Z) & \xrightarrow{\mathcal{F}(X)\otimes c_{Y,Z}} & \mathcal{F}(X)\otimes\mathcal{F}(Y)\otimes\mathcal{F}(Z) \\ {\scriptstyle m_{X,Y\otimes Z}}\downarrow & & \downarrow{\scriptstyle m_{X,Y}\otimes\mathcal{F}(Z)} \\ \mathcal{F}(X\otimes Y\otimes Z) & \xrightarrow[c_{X\otimes Y,Z}]{} & \mathcal{F}(X\otimes Y)\otimes\mathcal{F}(Z) \end{array} \qquad \begin{array}{ccc} \mathcal{F}(X\otimes Y)\otimes\mathcal{F}(Z) & \xrightarrow{c_{X,Y}\otimes\mathcal{F}(Z)} & \mathcal{F}(X)\otimes\mathcal{F}(Y)\otimes\mathcal{F}(Z) \\ {\scriptstyle m_{X\otimes Y,Z}}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(X)\otimes m_{Y,Z}} \\ \mathcal{F}(X\otimes Y\otimes Z) & \xrightarrow[c_{X,Y\otimes Z}]{} & \mathcal{F}(X)\otimes\mathcal{F}(Y\otimes Z) \end{array} \quad . \tag{2.3}$$

Definition 2.1. *Let* C *be a strict monoidal category. The strict monoidal category* $\mathsf{L}(\mathsf{C})$ *is defined as follows:*

Objects *We let* $\mathrm{Obj}(\mathsf{L}(\mathsf{C}))$ *be the free monoid on* $\{\underline{x} \mid x \in \mathrm{Obj}\,\mathsf{C}\}$. *It consists of finite lists of symbols of the form* $\underline{x}$, *for* $x \in \mathrm{Obj}\,\mathsf{C}$, *and the monoidal unit is the empty list,* $\emptyset$.

Morphisms *For* $x, y, z \in \mathrm{Obj}\,\mathsf{C}$, *we add generating morphisms* $\{\underline{f} : \underline{x} \to \underline{y} \mid f \in \mathrm{Hom}(x, y)\}$ *and a further generator* $\ell_{x,z} : \underline{x} \otimes \underline{z} \to \underline{x \otimes z}$. *Additionally, a generating morphism* $j : \emptyset \to \underline{\mathbb{1}}$. *These generators are subject to the following relations:*

(0) $\underline{\mathrm{id}_x} = \mathrm{id}_{\underline{x}}$;
(1) $\underline{g} \circ \underline{f} = \underline{g \circ f}$;
(2) $\ell_{y,y'} \circ \underline{f} \otimes \underline{f'} = \underline{f \otimes f'} \circ \ell_{x,x'}$ *for* $\underline{f} : \underline{x} \to \underline{y}$ *and* $\underline{f'} : \underline{x'} \to \underline{y'}$;
(3) $\ell_{x\otimes z,w} \circ (\ell_{x,z} \otimes \mathrm{id}_{\underline{w}}) = \ell_{x,z\otimes w} \circ (\mathrm{id}_{\underline{x}} \otimes \ell_{z,w})$;
(4) $\ell_{x,\mathbb{1}} \circ (\mathrm{id}_{\underline{x}} \otimes j) = \mathrm{id}_{\underline{x}}$;
(5) $\ell_{\mathbb{1},x} \circ (j \otimes \mathrm{id}_{\underline{x}}) = \mathrm{id}_{\underline{x}}$.

Let C, D be strict monoidal categories. We denote by $\mathbf{Lax}(\mathsf{C}, \mathsf{D})$ the category of lax monoidal functors from C to D, and by $\mathbf{Strict}(\mathsf{C}, \mathsf{D})$ the category of strict monoidal functors. In both cases, the morphisms are monoidal transformations, i.e. transformations $\sigma : \mathcal{F} \Rightarrow \mathcal{G}$ satisfying $\sigma_{X\otimes Y} \circ m^{\mathcal{F}}_{X,Y} = m^{\mathcal{G}}_{X,Y} \circ (\sigma_X \otimes \sigma_Y)$. Define $\mathbf{Oplax}(\mathsf{C}, \mathsf{D})$ similarly in terms of oplax monoidal functors, and $\mathbf{Frob}(\mathsf{C}, \mathsf{D})$ as the category of Frobenius monoidal functors. In this last case, we take as morphisms the transformations that are simultaneously morphisms of lax and oplax monoidal functors.

Proposition 2.2. *For strict monoidal categories* C, D*, there is an isomorphism of categories* $\mathbf{Lax}(\mathsf{C}, \mathsf{D}) \cong \mathbf{Strict}(\mathsf{L}(\mathsf{C}), \mathsf{D})$.

Proof. By definition of $\mathsf{L}(\mathsf{C})$, a strict monoidal functor $\mathcal{F} : \mathsf{L}(\mathsf{C}) \to \mathsf{D}$ is determined by a morphism of monoids $\mathcal{F}_0 : \mathrm{Obj}(\mathsf{L}(\mathsf{C})) \to \mathrm{Obj}(\mathsf{D})$, equivalently a function $\overline{\mathcal{F}_0} : \mathrm{Obj}(\mathsf{C}) \to \mathrm{Obj}(\mathsf{D})$, and by assignments of morphisms $\mathcal{F}(\underline{g}) : \mathcal{F}(\underline{x}) \to \mathcal{F}(\underline{y})$ for any $x, y \in \mathrm{Obj}\,\mathsf{C}$ and any $g \in \mathrm{Hom}_{\mathsf{C}}(x, y)$, as well as $\mathcal{F}(\ell_{x,z}) : \mathcal{F}(\underline{x} \otimes \underline{z}) = \mathcal{F}(\underline{x}) \otimes \mathcal{F}(\underline{z}) \to \mathcal{F}(\underline{x \otimes z})$ and $\mathcal{F}(\jmath) : \mathcal{F}(\emptyset) = \mathbb{1}_{\mathsf{D}} \to \mathcal{F}(\underline{\mathbb{1}_{\mathsf{C}}})$.

Prior to comparing axioms, we now observe that the above data coincides with the data required to specify a lax monoidal functor $\overline{\mathcal{F}} : \mathsf{C} \to \mathsf{D}$, defined by $\overline{\mathcal{F}}_0 := \overline{\mathcal{F}_0}$ and by $\overline{\mathcal{F}}(g) := \mathcal{F}(\underline{g})$ for any morphism g of C, with candidate lax monoidal structure afforded by $m^{\overline{\mathcal{F}}}_{x,z} := \mathcal{F}(\ell_{x,z})$ and $u^{\overline{\mathcal{F}}} := \mathcal{F}(\jmath)$. Items 0 and 1 are equivalent to the functoriality of $\overline{\mathcal{F}}$; item 2 is equivalent to the naturality of the candidate lax monoidal structure; item 3 is equivalent to axiom (2.1) for lax monoidal functors, and items 4 and 5 are equivalent to the respective unitality axioms (2.2) for such functors. The bijection between objects of $\mathbf{Lax}(\mathsf{C}, \mathsf{D})$ and $\mathbf{Strict}(\mathsf{L}(\mathsf{C}), \mathsf{D})$ follows.

Similarly, a monoidal transformation $\sigma : \mathcal{F} \Rightarrow \mathcal{G}$ in $\mathbf{Strict}(\mathsf{L}(\mathsf{C}), \mathsf{D})$ satisfies $\sigma_{\underline{x} \otimes \underline{z}} = \sigma_{\underline{x}} \otimes \sigma_{\underline{z}}$; hence, it is determined by the components of the form $\sigma_{\underline{x}}$, using which we define a transformation $\overline{\sigma} : \overline{\mathcal{F}} \Rightarrow \overline{\mathcal{G}}$, by setting $\overline{\sigma}_x = \sigma_{\underline{x}}$. Monoidality of $\overline{\sigma}$ is equivalent to the naturality square for σ commuting for the morphisms $\ell_{x,z}$ and $\jmath$, establishing bijections on Hom-sets. It is easy to verify the functoriality of these assignments. □

Remark 2.3. *Using the language of [10], we conclude that the topological theories on* C *are precisely the TQFTs on* $\mathsf{L}(\mathsf{C})$.

Proposition 2.4. *Let* C *be a strict monoidal category. Then* $\mathsf{L}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$ *is an oplax morphism classifier for* C*. In other words, for any strict monoidal category* D *we find an isomorphism of categories* $\mathbf{Oplax}(\mathsf{C}, \mathsf{D}) \cong \mathbf{Strict}(\mathsf{L}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}, \mathsf{D})$.

Proof. $\mathbf{Oplax}(\mathsf{C}, \mathsf{D}) \simeq \mathbf{Lax}(\mathsf{C}^{\mathrm{op}}, \mathsf{D}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathbf{Strict}(\mathsf{L}(\mathsf{C}^{\mathrm{op}}), \mathsf{D}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathbf{Strict}(\mathsf{L}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}, \mathsf{D})$. □

Corollary 2.5. *The oplax classifier* $\mathsf{opL}(\mathsf{C})$ *can be presented analogously to* $\mathsf{L}(\mathsf{C})$*, involving generators* $\hbar_{x,z} : \underline{x \otimes z} \to \underline{x} \otimes \underline{z}$ *rather than* $\ell_{x,z}$ *and* $q : \underline{\mathbb{1}} \to \emptyset$ *rather than* $\jmath$.

The previous corollary is easy to see with diagrams; see (3.4) and (3.5).

Definition 2.6. *The Frobenius classifier* $\mathsf{F}(\mathsf{C})$ *is defined as follows. We let* $\mathrm{Obj}(\mathsf{F}(\mathsf{C}))$ *be the free monoid on* $\{\underline{x} \mid x \in \mathrm{Obj}\,\mathsf{C}\}$*. For all* $x, y \in \mathsf{C}$*, we add generating morphisms* $\{\underline{f} : \underline{x} \to \underline{y} \mid f \in \mathrm{Hom}(x, y)\}$ *and further generators* $\ell_{x,z} : \underline{x} \otimes \underline{z} \to \underline{x \otimes z}$ *and* $\hbar_{x,z} : \underline{x \otimes z} \to \underline{x} \otimes \underline{z}$*. Finally, we add generating morphisms* $\jmath : \emptyset \to \underline{\mathbb{1}}$ *and* $q : \underline{\mathbb{1}} \to \emptyset$.

We impose all the relations of Definition 2.1, as well as "oppositized" variants of items 2 to 5 for $\hbar_{x,x'}$ *and* q*, following Corollary 2.5. Additionally, we impose the relations*

$$\hbar_{x \otimes z, w} \circ \ell_{x, z \otimes w} = (\ell_{x,z} \otimes \mathrm{id}_{\underline{w}}) \circ (\mathrm{id}_{\underline{x}} \otimes \hbar_{z,w}) \tag{2.4}$$

and

$$\hbar_{x, z \otimes w} \circ \ell_{x \otimes z, w} = (\mathrm{id}_{\underline{x}} \otimes \ell_{z,w}) \circ (\hbar_{x,z} \otimes \mathrm{id}_{\underline{w}}). \tag{2.5}$$

Theorem 2.7. *For strict monoidal categories* C, D*, there is an isomorphism of categories* $\mathbf{Frob}(\mathsf{C}, \mathsf{D}) \cong \mathbf{Strict}(\mathsf{F}(\mathsf{C}), \mathsf{D})$.

Proof. Let $\mathcal{F} : \mathsf{F(C)} \longrightarrow \mathsf{D}$ be a strict monoidal functor. Similar to the proof of Proposition 2.2, the assignments $\overline{\mathcal{F}}(x) = \mathcal{F}(\underline{x})$, $\overline{\mathcal{F}}(f) = \mathcal{F}(\underline{f})$ define a functor $\overline{\mathcal{F}} : \mathsf{C} \longrightarrow \mathsf{D}$, and the maps $\mathcal{F}(\ell_{x,z})$ and $\mathcal{F}(j)$ define a lax monoidal structure on $\overline{\mathcal{F}}$. Following Corollary 2.5, the maps $\mathcal{F}(k_{x,z})$ and $\mathcal{F}(q)$ define an oplax monoidal structure on $\overline{\mathcal{F}}$. The relations (2.4, 2.5) are equivalent to the axiom (2.3) making $\overline{\mathcal{F}}$ a Frobenius monoidal functor. Also the correspondence for natural transformations extends similarly to the proof of Proposition 2.2. □

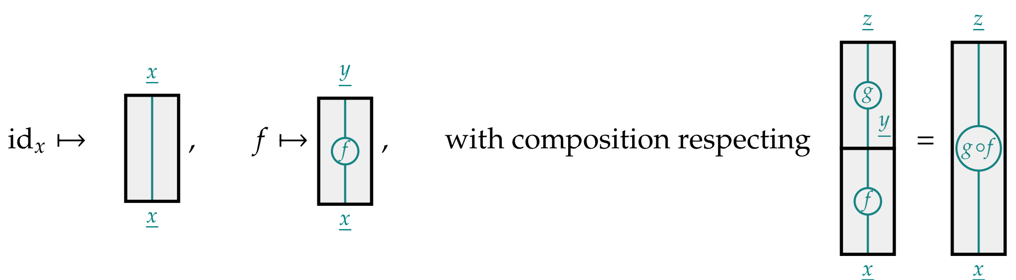
Let $\mathcal{E} : \mathsf{C} \longrightarrow \mathsf{F(C)}$ be the functor corresponding to $\mathrm{Id}_{\mathsf{F(C)}}$ under the correspondence of Theorem 2.7. The following is very easy to verify.

Lemma 2.8. *Given a Frobenius monoidal functor $\mathcal{F} : \mathsf{C} \longrightarrow \mathsf{D}$, we have $\mathcal{F} = \overline{\mathcal{F}} \circ \mathcal{E}$.*

3. Diagrammatic interpretation

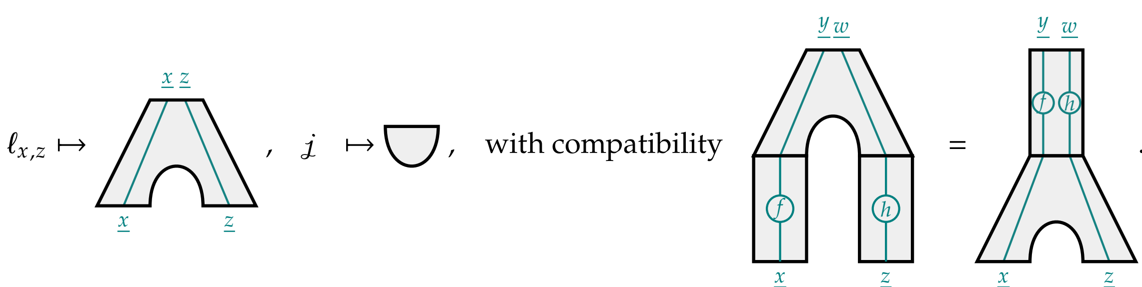
The generators and relations for the classifiers $\mathsf{L(C)}$, $\mathsf{opL(C)}$ and $\mathsf{F(C)}$ defined in Section 2 can be interpreted using diagrams very similar to those of [17, 21, 22, 20]. More precisely, we interpret the classifier by enclosing the string calculus of the monoidal category inside an envelope.

We now describe the diagrammatics. For all $x, y, z, w \in \mathsf{C}$, $f \in \mathrm{Hom}_{\mathsf{C}}(x, y)$, $g \in \mathrm{Hom}_{\mathsf{C}}(y, z)$, $h \in \mathrm{Hom}_{\mathsf{C}}(z, w)$, we associate



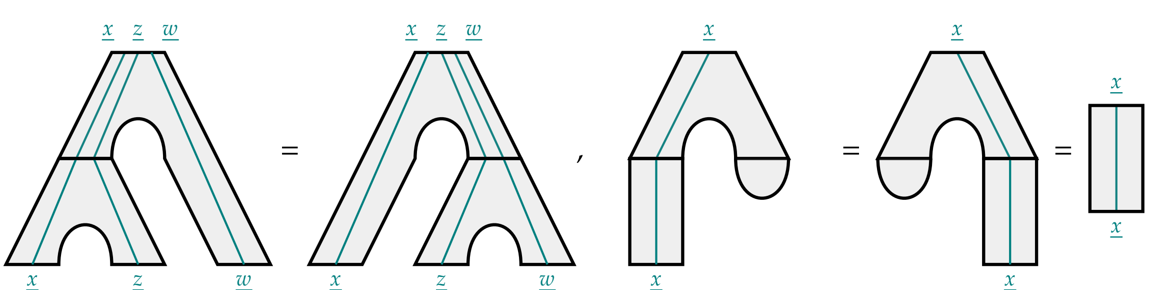
(3.1)

The extra morphisms of $\mathsf{L(C)}$ are given diagrammatically by the following diagrams for $x, z \in \mathrm{Obj(C)}$



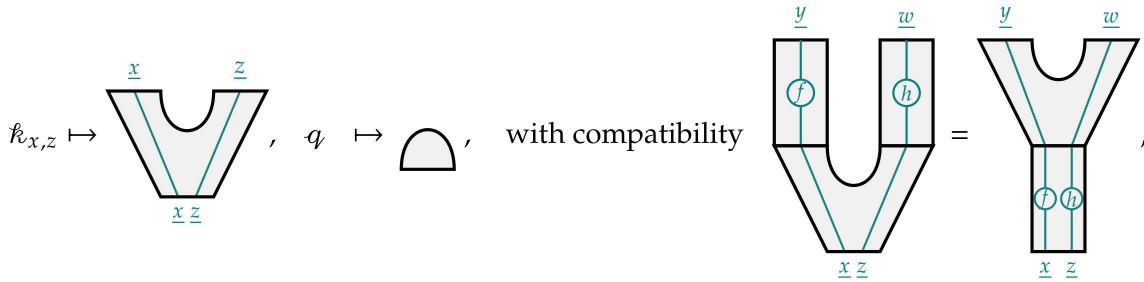
(3.2)

The relations for $\mathsf{L(C)}$ correspond to the following identities:



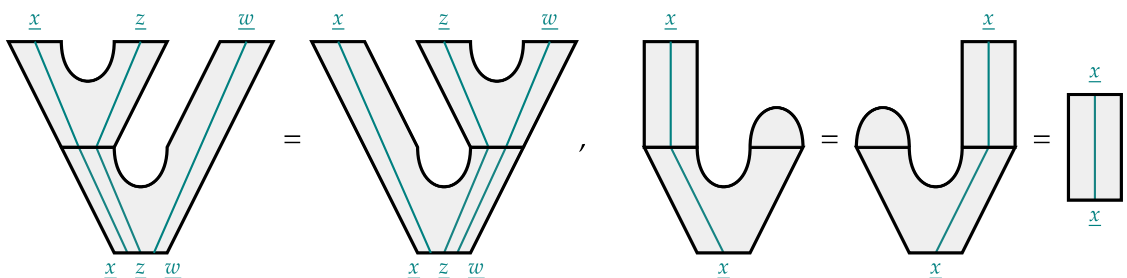
(3.3)

For $\mathsf{opL}(\mathsf{C})$ we have instead the diagrams



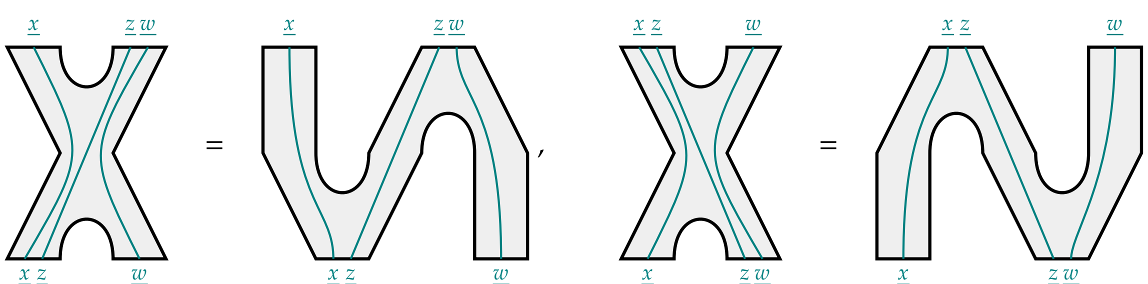
(3.4)

and the relations correspond to the identities reversed from (3.3), reproduced below for convenience,



(3.5)

Finally, for $\mathsf{F}(\mathsf{C})$ we have both (3.2) and (3.4) with relations (3.3) and (3.5) augmented by the following Frobenius compatibility relations

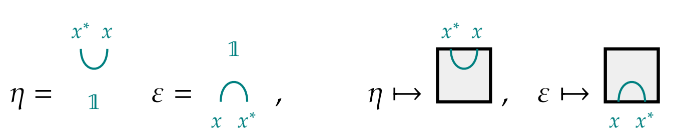


(3.6)

Remark 3.1. *Mulevičius considered a similar graphical calculus for monoidal functors, embedding string calculus in cylindrical "tubes" in the context of ribbon Frobenius functors [20, Fig. 4.1 (F1–F3)]; see also Ponto–Schulman for lax symmetric monoidal functors [21, Fig. 7]. Mulevičius' calculus also allows for braided and ribbon Frobenius functors.*

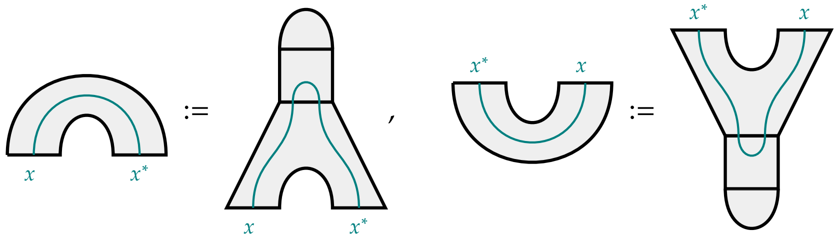
Lemma 3.2. *Let x be a right rigid object of C and let x^* be a rigid right dual of x. Then $\underline{x^*}$ is a rigid right dual of $\underline{x}$ in $\mathsf{F}(\mathsf{C})$.*

Proof. Let $\eta : \mathbb{1} \longrightarrow x^* \otimes x$ and $\varepsilon : x \otimes x^* \longrightarrow \mathbb{1}$ be the unit and counit for the duality, which we denote in the string calculus of C and in the diagrammatics of $\mathsf{F}(\mathsf{C})$ as



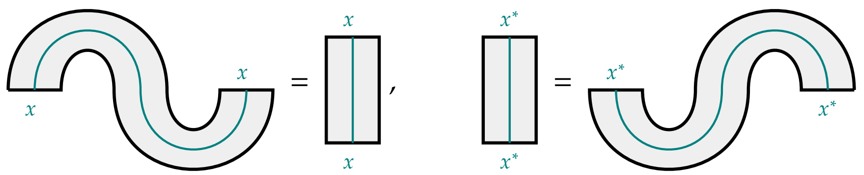
(3.7)

Then the morphisms



(3.8)

satisfy the zigzag equations



(3.9)

□

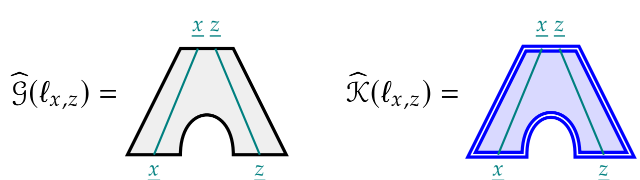
Corollary 3.3. *[6, Theorem 2] A Frobenius monoidal functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ preserves dual pairs.*

Proof. Using the functor $\mathcal{E} : \mathsf{C} \to \mathsf{F}(\mathsf{C})$ of Lemma 2.8, we write $\mathcal{F} = \overline{\mathcal{F}} \circ \mathcal{E}$. Since $\overline{\mathcal{F}}$ is strict monoidal, it preserves dual pairs, and $\mathcal{E}$ preserves dual pairs by Lemma 3.2. □

As an example of application, we give the diagrammatic proof that, for Frobenius monoidal functors, a lax and oplax monoidal transformation between them is invertible. The diagrammatic proof was given in a talk by McCurdy [18]; see also [6, Prop. 7], [21, Prop. 2.10] for the statement.

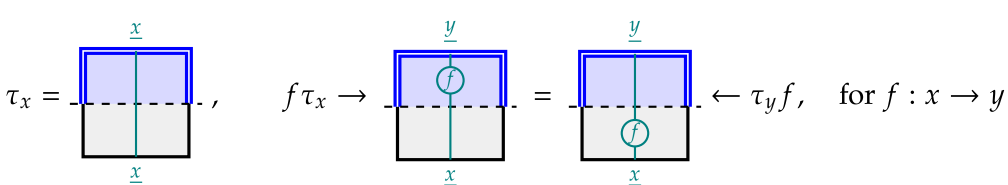
Proposition 3.4. *Given Frobenius monoidal functors $\mathcal{G}, \mathcal{K} : \mathsf{C} \to \mathsf{D}$ for C rigid, if $\tau : \mathcal{G} \to \mathcal{K}$ is a lax and oplax monoidal transformation, then τ is invertible.*

Proof. We do a diagrammatic proof to construct the inverse of τ. We draw the diagrams of the functor $\mathcal{G}$ as we did previously, and add colours and double lines to distinguish those of $\mathcal{K}$



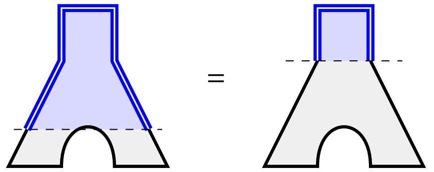
(3.10)

We represent the natural transformation τ_x as a dotted line, and its naturality is expressed diagrammatically by allowing crossing:



(3.11)

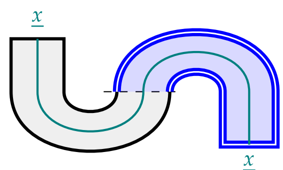
The fact that τ is lax monoidal expresses itself diagrammatically as



(3.12)

We construct the inverse via the following diagrammatic construction using the rigidity of C (3.8)

$\tau_x^{-1} =$



(3.13)

□

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