A Howe dual pair in Weyl–Clifford algebras

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Context

We study an algebra $\mathfrak{O}_d \subset \mathcal{W}_d \otimes \mathcal{C}l(V)$ centralising a realisation of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$. The algebra $\mathfrak{O}$ is called the *Total angular momentum algebra* [Langlois-Rémillard 2023]. A set of generators is exhibited in [De Bie, Oste, et al. 2018] and advances have recently been made to find all their relations [Oste 2021; Calvert et al. 2025], but the proof is not yet completed for higher dimensions.
Using a maximal abelian subalgebra $\mathfrak{Y} \subset \mathfrak{O}_d$, we can study the representation theory of the algebra using a weight structure and exhibiting elements related to classical ladder operators and their factorisations.

Weyl–Clifford algebra and the Howe dual pair $(\mathfrak{osp}(1|2) \times \mathsf{Spin}(d))$

The Clifford algebra $\mathcal{C}l(d)$ has generators $e_1, \dots, e_d$ such that $\{e_j, e_k\} = 2\delta_{jk}$.

$$\mathcal{W}_d = \langle x_1, \dots x_n, \partial_{x_1}, \dots, \partial_{x_n} \mid [x_j, x_k] = 0 = [\partial_{x_j}, \partial_{x_k}], [\partial_{x_j}, x_k] = \delta_{jk} \rangle$$

There is a realisation of $\mathfrak{osp}(1|2)$ in $\mathcal{W}_d \otimes \mathcal{C}l(d)$, Howe dual with $\mathsf{Spin}(d)$.

The monogenic representation

The Dirac operator and its dual symbol are element of $\mathcal{W} \otimes \mathcal{C}l(d)$

$$\underline{D} := \sum_{j=1}^{d} \partial_{x_j} \otimes e_j, \qquad \underline{x} := \sum_{j=1}^{d} x_j \otimes e_j$$

are the generators of an $\mathfrak{osp}(1|2)$-realisation. The polynomial null-solutions of the Dirac equation

$$\underline{D}p = 0$$

of degree n are an irreducible representation for the algebra of symmetry of this $\mathfrak{osp}(1|2)$-realisation. We call it the **monogenic representation** of degree n.
We can express the solution p via special functions or via generalised symmetries (also in the Dunkl case) [De Bie, Langlois-Rémillard, et al. 2023]

$$\mathfrak{z}_j := x_j H - \underline{x}\partial_{x_j}\underline{x}.$$

Corollary $(d = 4)$ For $d = 1, 2, 3, 4$, any finite-dimensional irreducible representation of $\mathfrak{O}$ is isomorphic to the monogenic representation.

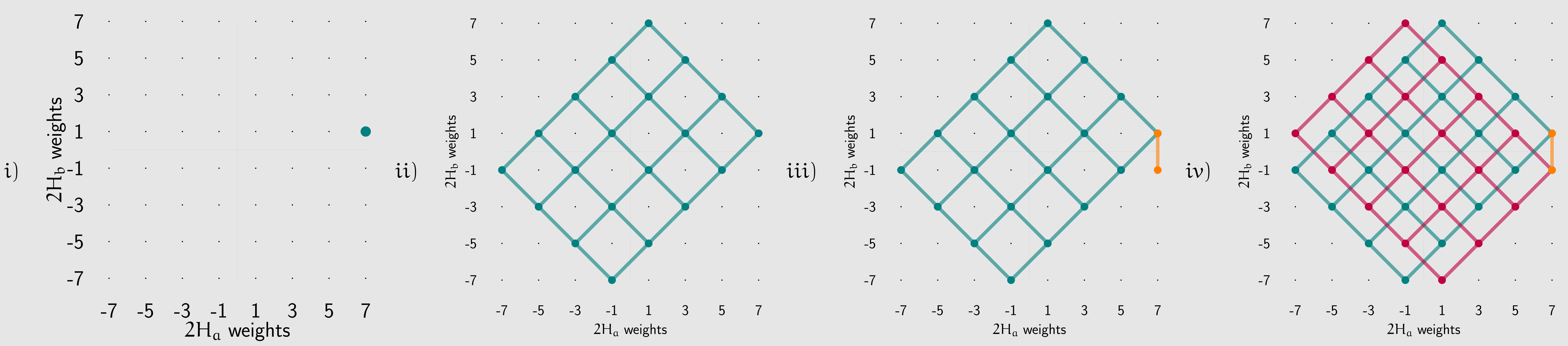
Main result and conjecture [Kafando 2026+]

Conjecture. Let V be a finite-dimensional irreducible representation of $\mathfrak{O}_d$. The representations are indexed by an integer N and are isomorphic to monogenic representation

Theorem $(d = 4)$. The finite-dimensional irreducible representations of $\mathfrak{O}_4$ are given by a family of representations $M(N)$ for $N \in \mathbb{N}$. Let v be its highest-weight with respect to $\mathfrak{Y}$. The spectrum of $\mathfrak{Y}$ is given by

$$H_1 v = (N + \tfrac{1}{2})v, \qquad H_2 v = \tfrac{1}{2}v, \qquad Zv = -(N + \tfrac{3}{2})v.$$

Construction of the representation



Basis and Maximal abelian subalgebra $(d = 4)$

Highest-weight vector v

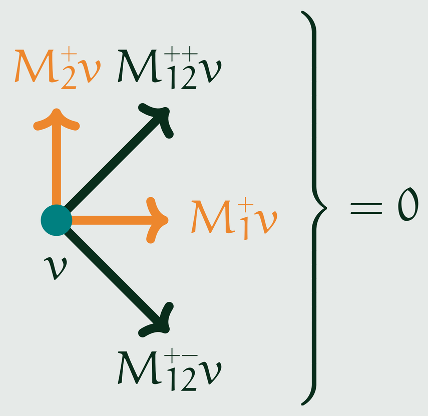
$$\mathfrak{Y}_4 = \langle H_1, H_2, Z \rangle \subset \mathfrak{O}_4 = \langle H_1, H_2, M_1^{\pm}, M_2^{\pm}, M_{12}^{\pm\pm}, M_{12}^{\pm\mp} \rangle$$

$$\mathcal{B}_N = \{ v_{jk}^{\epsilon} = (M_{12}^{-\epsilon})^j (M_{12}^{-\bar{\epsilon}})^k (M_2^-)^{\epsilon} v \mid j \in \{0, \dots, N\}, k \in \{0, \dots, N+1\}, \epsilon \in \{0, 1\} \}$$

Characterisation of representations $(d = 4)$

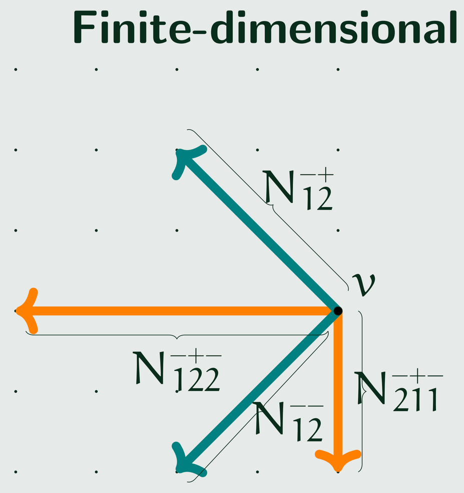
Let v be a highest-weight vector with (H_1, H_2)-weights (λ_1, λ_2) and central weight Λ.
We have ladder operators $M_{12}^{++}, M_{12}^{+-}, M_{12}^{-+}, M_{12}^{--}$, and $M_a^+, M_a^-, M_b^+, M_b^-$.

Lowest weight



$$M_1^- M_1^+ v = 0,$$
$$M_2^- M_2^+ v = 0,$$
$$M_{12}^{--} M_{12}^{++} v = 0,$$
$$M_{12}^{-+} M_{12}^{+-} v = 0,$$

Finite-dimensional



$$(M_{12}^{-+})^{N_{12}^{-+}+1} v = 0$$
$$(M_{12}^{--})^{N_{12}^{--}+1} v = 0$$
$$(M_1^-)^{N_1^-+1} v = 0$$
$$(M_2^-)^{N_2^-+1} v = 0$$

Lemma: The factorisations $M_a^{\pm} M_a^{\mp}, M_{12}^{\pm\pm} M_{12}^{\mp\mp}, M_{12}^{\pm\mp} M_{12}^{\mp\pm}$ are known and are in $\mathfrak{Y}$, e.g.

$$M_{ab}^{-+} M_{ab}^{+-} = 16((H_a + 1/2)^2)((H_b - 1/2)^2)((H_a - H_b + 1)^2 - (O_{1234} + 1/2)^2)$$

The factorisations are used to prove the classification theorem.

Future

- Extend to all dimensions (Kafando–Langlois-Rémillard–Nonkané)
- Have a full description of the ideal of relations conjectured in Calvert et al. 2025 (Langlois-Rémillard–Papageorgiou-Kafka, Calvert–De Martino)
- Extend to rational Cherednik algebras [Oste 2021] (Oste–Langlois-Rémillard)

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