Weaving weights: double dihedral deformation

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Main result in two sentences

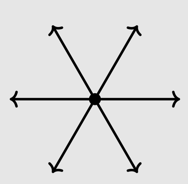
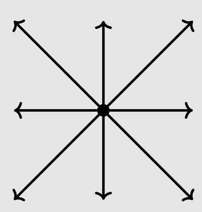
The process described below classifies a class of finite-dimensional irreducible representations of a deformation of the total angular momentum algebra by exhibiting a tuple of parameters and giving the constraints they need to obey. The focus is put on dimension four with the deformation occurring by means of two dihedral groups, with generalisations to any number of dihedral groups in mind.

Context: Symmetry algebra of an $\mathfrak{osp}(1|2)$ realisation

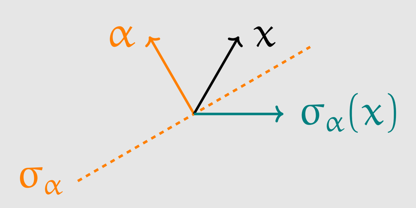
We study an algebra $\mathfrak{S}(W, V, \kappa) \subset \mathfrak{h}_\kappa(W) \otimes \mathcal{Cl}(V)$ consisting in the centraliser of the $\mathfrak{osp}(1|2)$ realisation inside the tensor product of a rational Cherednik algebra and a Clifford algebra for a reflection group W. The representation theory of the algebra has been studied for a few groups ($W = \mathbb{Z}_2^d$ [De Bie et al. 2016] and $W = D_{2m} \times \mathbb{Z}_2$ [De Bie et al. 2022]).

We focus on $W = D_{2m} \times D_{2n} \subset O(4)$. We characterize finite-dimensional representations of $\mathfrak{S}(W, V, \kappa)$ constructed via restriction to a subalgebra $\mathfrak{T} \subset \mathfrak{S}(W, V, \kappa)$ admitting a triangular structure. Then, we give the constraints that the parameters of the construction must satisfy.

Dihedral groups and their root systems

D_6 :  D_8 : 

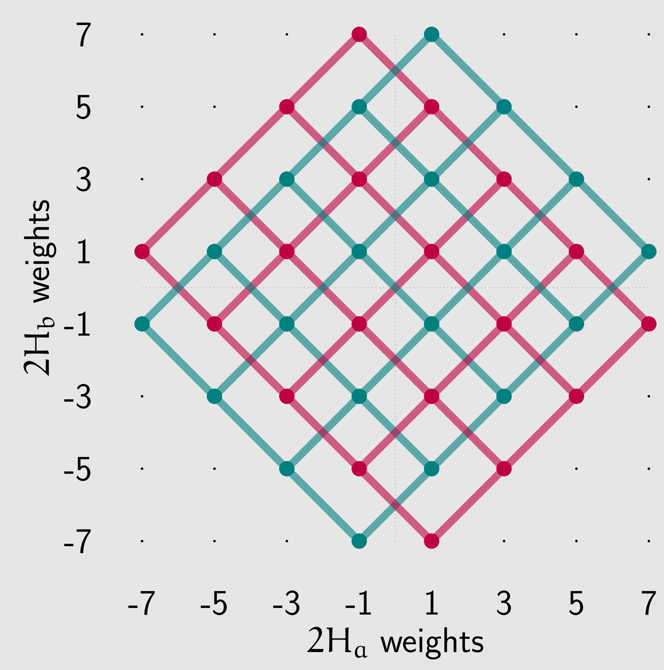
$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha$$



Classical situation: total angular momentum algebra

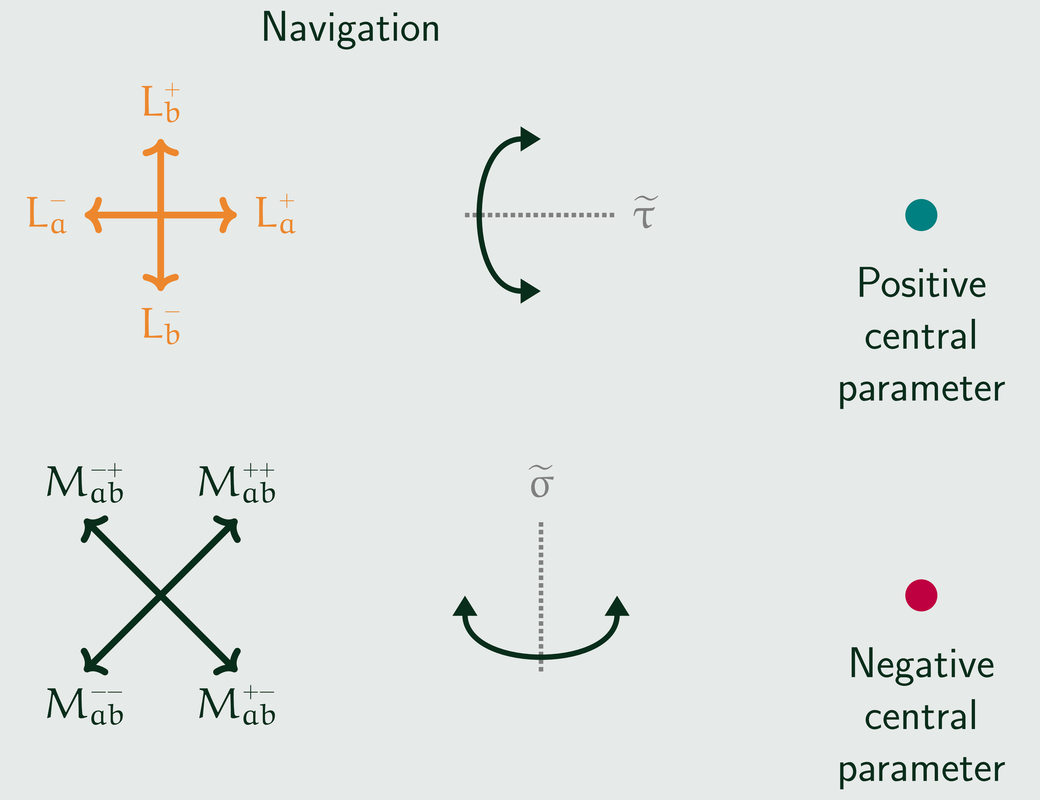
In the classical case, we find an algebra inside $\mathfrak{S}$ realising $\mathfrak{so}(d)$. For $d = 4$, we have an exceptional isomorphism that translate to the weights:

$$\mathfrak{so}(4) \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2).$$

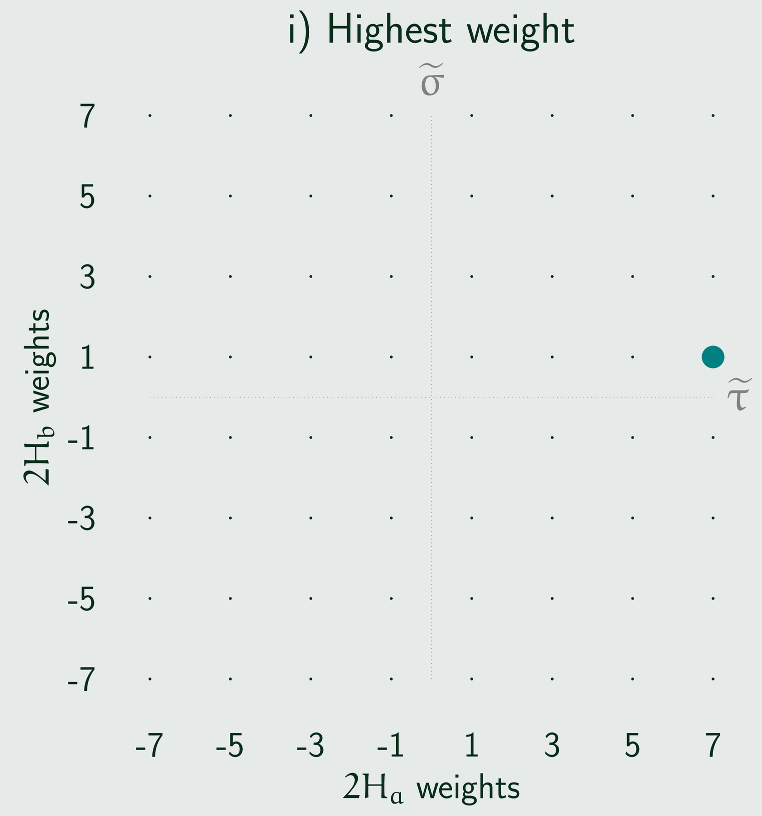


A tapestry of irreducible representations: weaving the weight spaces

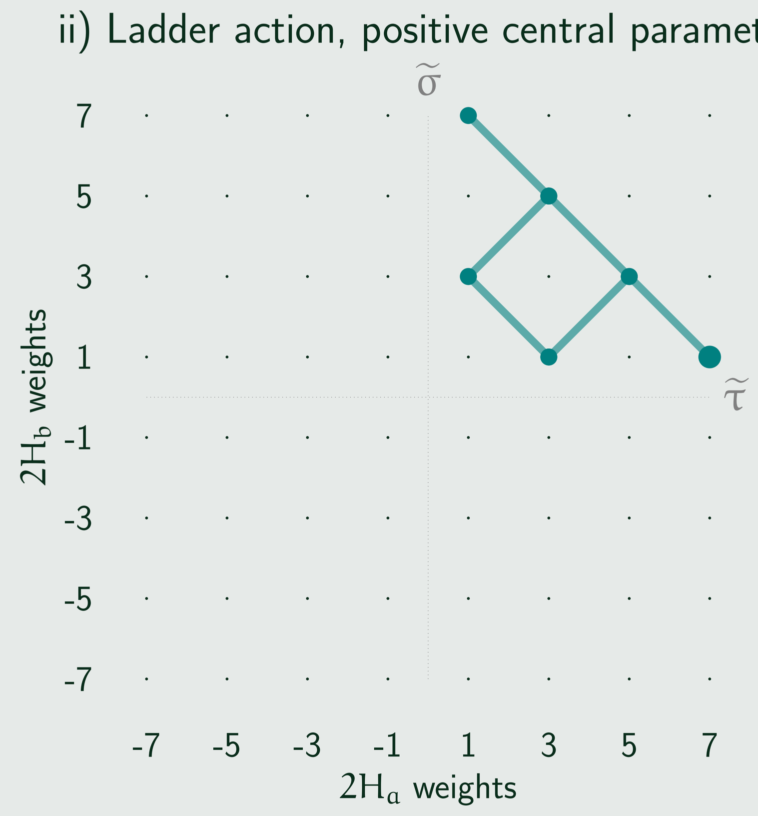
Navigation



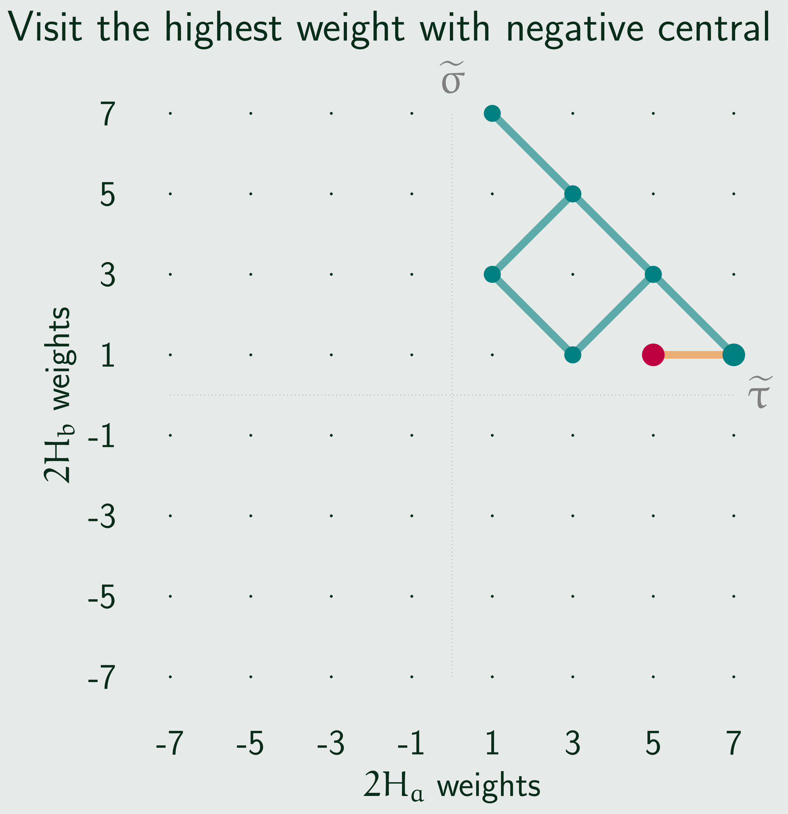
i) Highest weight



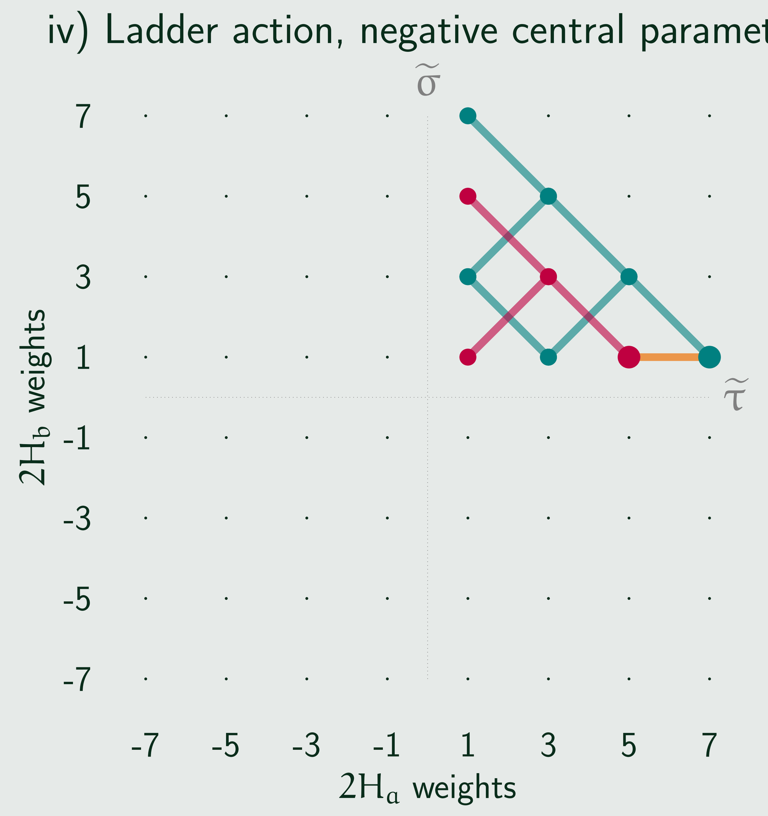
ii) Ladder action, positive central parameter



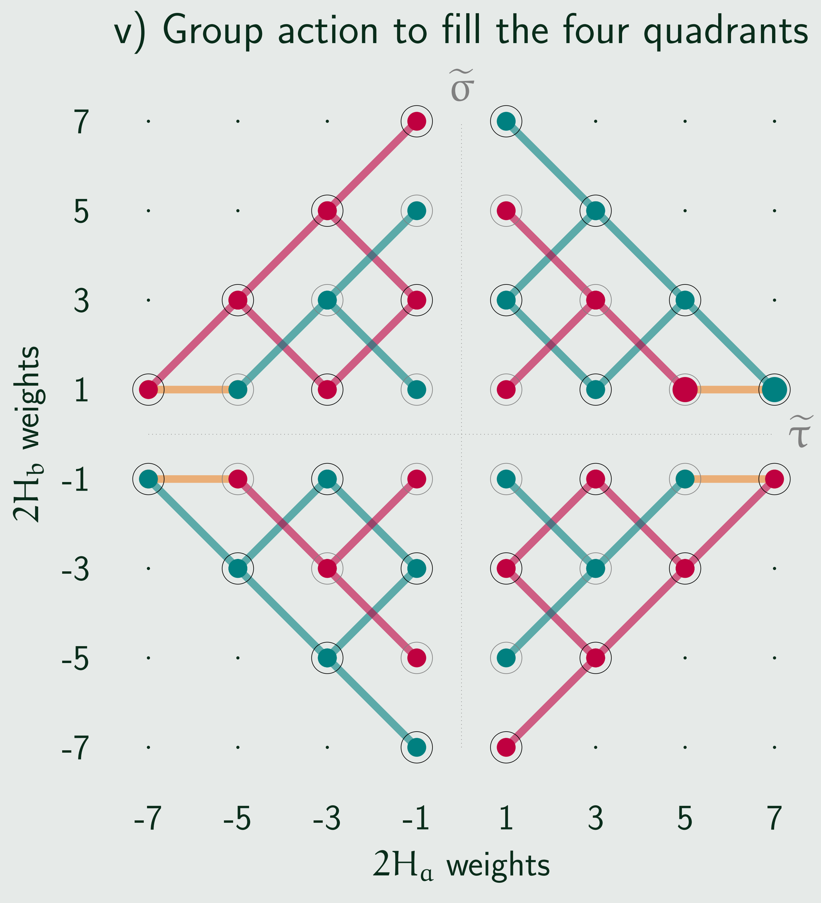
iii) Visit the highest weight with negative central parameter



iv) Ladder action, negative central parameter



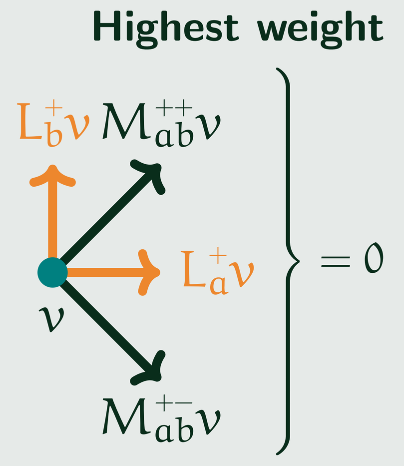
v) Group action to fill the four quadrants



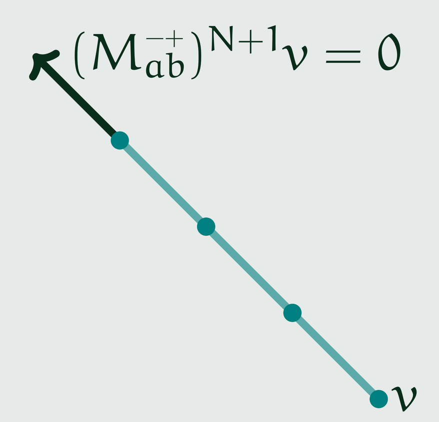
Characterisation of the representations

Let v be the highest weight vector with (H_a, H_b)-weights (λ_a, λ_b) and central parameter Λ. It lies in $\widetilde{U} \in \mathrm{Irr}(\widetilde{W})$. We have ladder operators $M_{ab}^{++}, M_{ab}^{+-}, M_{ab}^{-+}, M_{ab}^{--}$, and $L_a^+, L_a^-, L_b^+, L_b^-$.

Highest weight



Finite-dimensional



$$L_a^-L_a^+v = 0, \quad L_b^-L_b^+v = 0, \quad M_{ab}^{--}M_{ab}^{++}v = 0, \quad M_{ab}^{-+}M_{ab}^{+-}v = 0, \quad M_{ab}^{+-}M_{ab}^{-+}(M_{ab}^{-+})^N v = 0.$$

It translates to a system of equations, with $F_{a,i}^{\pm}$, $F_{b,i}^{\pm}$ given from $\widetilde{U}$:

$$((\lambda_a + 1/2)^2 - F_{a,0}^-)((\Lambda + \lambda_b)^2 - (\lambda_a + 1/2)^2) = 0,$$
$$((\lambda_b + 1/2)^2 - F_{b,0}^-)((\Lambda + \lambda_a)^2 - (\lambda_b + 1/2)^2) = 0,$$
$$((\lambda_a + 1/2)^2 - F_{a,0}^-)((\lambda_b + 1/2)^2 - F_{b,0}^-)((\lambda_a + \lambda_b + 1)^2 - (\Lambda - 1/2)^2) = 0,$$
$$(\lambda_a + 1/2)^2 - F_{a,0}^-)((\lambda_b - 1/2)^2 - F_{b,0}^+)((\lambda_a - \lambda_b + 1)^2 - (\Lambda + 1/2)^2) = 0,$$
$$((\lambda_a - N - 1/2) - F_{a,N}^+)((\lambda_b + N + 1/2)^2 - F_{b,N}^-)((\lambda_a - \lambda_b - 2N - 1)^2 - (\Lambda + 1/2)^2) = 0.$$

An example: Null solutions of the Dunkl–Dirac operator

Let $W \subset O(d)$ be a reflection group with R its root system. The Dunkl operators [Dunkl 1989]

$$\mathcal{D}_i f := \partial_{x_i} + \sum_{\alpha \in R^+} \kappa(\alpha)\alpha_i \frac{f - \sigma_\alpha f}{\langle \alpha, -\rangle},$$

where $\kappa : R \to \mathbb{C}$ is a W-invariant function, generalise partial derivatives. Denote by $\mathcal{Cl}(d)$ the Clifford algebra with generators $e_1, \ldots, e_d$ respecting $\{e_j, e_k\} = 2\delta_{jk}$. The Dunkl–Dirac operator and its dual symbol

$$\underline{D} := \sum_{j=1}^{d} D_j e_j, \qquad \underline{x} := \sum_{j=1}^{d} x_j e_j$$

are the generators of an $\mathfrak{osp}(1|2)$-realisation. The polynomial solutions of degree n of the Dunkl–Dirac equation $\underline{D}p = 0$ form an irreducible representation for the symmetry algebra of this $\mathfrak{osp}(1|2)$-realisation. They are examples of the representations we construct.

References

- De Bie, H., A. Langlois-Rémillard, R. Oste, and J. Van der Jeugt (2022). In: *Journal of Algebra* 591, pp. 170–216. DOI: 10.1016/j.jalgebra.2021.09.025.
- De Bie, H., V. X. Genest, and L. Vinet (2016). In: *Advances in Mathematics* 303, pp. 390–414. DOI: 10.1016/j.aim.2016.08.007.
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