On (Iwahori)-Hecke algebras

with a view toward Double Affine Hecke algebras

Seminar transcript from
the Fall 2019 Kleine Seminar

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Introduction

Welcome to the *Kleine Seminar*! Introduced in the summer 2019 by a small gathering of (post)-doctoral students at Universiteit Gent, the Kleine Seminar is thought to be held during the course session and has the objective to study some subjects the members are interested in. For each subject, every member will get to present a piece of the material.

Subjects are chosen on a unanimous vote from the presenting member

This small book groups the minutes of the meetings based upon a member live-TEXing and the presenter own notes. Some comments in the margin are placed through the text to represent the discussion, formal and informal, the members had. It is based loosely in the graffiti the interesting Concrete Mathematics book by Graham and Knuth [3]. Some of them are signed if the writer remembers (and deem the correct worthy of authorship).

It's not really mandatory, but it does add somewhat a more discussing tone!

The Fall 2019 session aims to uncover the secrets of Double Affine Hecke Algebras (DAHA) []. For this objective, the members deemed interesting to recover the classical (Iwahori)-Hecke algebras representation theory from Andrew Mathas's book [5] before attacking the Double Affine Hecke case. The DAHA is an unifying theme of the research of many amongst us and having in common this language would, hopefully, help collaboration between us.

At the time of writing this graffiti, I am still unsure about distinction between Iwahori-Hecke and Hecke and I think the parentheses reflect this general sentiment

An interesting point of Mathas approach is its use of cellular algebras. With strong links in diagrammatic algebras, this approach also should be interesting and even useful to some of your research. Finally, its emphasis on symmetric group offer an example to apply the theory in the A_{n-1} root system, something that appears in the research of many of the members.

All comments are welcomed, corrections can be directly made by one of the member, so contacting any of them should work, as long as they are still at Universiteit Gent.

Sincerely, the committee

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Asmus Bisbo;
Sam Claerebout;
Hadewijch De Clercq;
Alexis Langlois-Rémillard;
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Chapter 1

Introducing the (Iwahori)-Hecke algebras

Presented by Asmus Bisbo on 19-09-2019.
Notes recorded by Alexis Langlois-Rémillard and Wouter van de Vijver.

1.1 Symmetric group

In this section, the symmetric group is defined in a Coxeter group fashion and some technical lemmas are proved to prepare for the proof of the definition of the Iwahori-Hecke algebra. Most of the proof are quite technical and are all done in the chapter 1 of Mathas and shall thus be absent from these notes.

Notation Let $n \in \mathbb{N}$ and S_n the symmetric group with a right action on the set $\{1 \ldots, n\}$. Let i run from 1 to $n-1$ and $s_i := (i, i+1)$. Put $S = \{s_1, \ldots, s_{n-1}$ to be the set of simple transposition of the symmetric group.

As a Coxeter group the group S_n is generated by the s_i with relations:

Note that the last relation is equivalent to $(s_i s_{i+1})^3 = 1$. Al.

$$s_i^2 = 1;$$

$$\text{braid relations} \quad \begin{cases} s_i s_j = s_j s_i & 1 \leq i < j-1 \leq n-2; \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & 1 \leq i \leq n-2. \end{cases}$$

Let $\omega \in S_n$ and write $\omega = s_{i_1} \ldots s_{i_k}$. If k is minimal then ω has length k, noted by $\ell(\omega) = k$, and the presentation is said to be reduced.

Dyer reflection cocycle is defined by:

This Dyer cocyle has nice interpretation in my research on polynomial representations of $\mathfrak{osp}(1, 2n)$, especially for ordering Young tableaux and diagrams. As.

$$N(\omega) = \left\{(j, n) \in S_n \,|1 \leq j < k \leq n, \quad j\omega > k\omega\right\}.$$

Watchout for the dot on the plus!

Define $A \dot{+} B := A \cup B \backslash (A \cap B)$ to be the symmetric sum for two sets A and B.

Lemma 1.1.1 (Dyer)**.** *If $v, w \in S_n$ then $N(vw) = N(v) \dot{+} vN(w)v^{-1}$*

Proof. The proof is based on induction on the length of the permutation v. □

Here Wouter's live-TEX ended and we must rely on Alexis's manuscript notes, let's hope!

We introduce new notation. Let

$$T = \{(jk) \mid 1 \leq j < k \leq n\} = \bigcup_{\omega \in S_n} \omega S \omega^{-1}.$$

The proof is to pick a reduced expression, then t_a is to be the element of order 2 in the first a simple transpositions and use preceding lemma repeatedly to arrive at a decomposition in the t_a. Now prove $a \neq b$ implies $t_a \neq t_b$ by contradiction because it's reduced and the result is given. Al.

Proposition 1.1.2. *If $\omega \in S_n$, then*

1. *$\ell(\omega) = |N(\omega)|$;*
2. *$N(\omega) = \{t \in T \mid \ell(t\omega) < \ell(\omega)\}$.*

This proposition has a direct corollary.

Corollary 1.1.3. *Let $\omega \in S_n$ and $s_i \in S$.*

$$\ell(s_i \omega) = \begin{cases} \ell(\omega) + 1 & i\omega < (i+1)\omega; \\ \ell(\omega) - 1 & i\omega > (i+1)\omega. \end{cases}$$

Now is a good moment to recall that S_n acts on $\{1, \dots n\}$ on the right if you had forgotten, like I did! Al.

The following theorem is useful to prove the main theorem of this section.

Theorem 1.1.4 (Strong exchange condition)**.** *Let*

$$s_{i_1}, \dots s_{i_k} \in S, t \in T \text{ with } \ell(ts_{i_1} \dots s_{i_k}) < \ell(s_{i_1} \dots s_{i_k}).$$

The hat denote that the elements is missing. As.

Then,

$$ts_{i_1} \dots s_{i_k} = s_{i_1} \dots \hat{s_{i_a}} \dots s_{i_k}$$

and furthermore,

$$t = s_{i_1} \dots s_{i_{a-1}} s_{i_a} s_{i_{a-1}} \dots s_{i_k}.$$

The proof reproduce the argument of the previous proposition and uses Dyer's lemma.

An equivalence relation $\sim_b$ is given between reduced expressions $s_{i_1}, \dots, s_{i_k}$ and $s_{j_1}, \dots, s_{j_k}$ if it is possible to go from one to another using only braid relations. The next theorem is given to give a well defined basis for the Iwahori-Hecke algebras.

Theorem 1.1.5 (Matsumoto)**.** *Let $s_{i_1}, \dots, s_{i_k}$ and $s_{j_1}, \dots, s_{j_k}$ be two reduced expression. They are equivalent if and only if $s_{i_1} \dots s_{i_k} = s_{j_1} \dots s_{j_k}$.*

This theorem gives the "other way" of the definition.

The proof uses the strong exchange condition and induction in the length to proceed.

Remark that this theorem is really about Artin braid group, that is the group generated by a corresponding set of generators as S_n and satisfying the braid condition, but not that they square to the identity.

1.2 Iwahori-Hecke algebras

Wouter came back to TEXing now!

Definition 1.2.1 (Iwahori-Hecke algebra). . *Let R be a commutative domain and $q \in R$. Then $\mathscr{H}_{R,q(S_n)}$ is generated by $T_1, \dots, T_{n+1}$ satisfying*

According to common definition, it should be a ring with no zero divisor. Note also that in the book, Mathas ask for a unit.

$$\begin{aligned} &(T_i - q)(T_i + 1) = 0 && i = 1 \dots n-1 \\ &T_iT_j = T_jT_i && 1 \le i < j-1 \le n-2 \\ &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} && i = 1 \dots n-2. \end{aligned}$$

This should really be viewed as a q-deformation of the symmetric group. Al.

Lemma 1.2.2. *Let $s \in S$ and $\omega \in S_n$. Then*

$$T_\omega T_s = \begin{cases} T_{\omega s} & \text{if } \ell(\omega s) > \ell(\omega) \\ qT_{\omega s} + (q-1)T_\omega & \text{if } \ell(\omega s) < \ell(\omega) \end{cases} \tag{1.1}$$

Furthermore we have an equivalent statement for

$$T_s T_\omega = \begin{cases} T_{s\omega} & \text{if } \ell(s\omega) > \ell(\omega) \\ qT_{s\omega} + (q-1)T_\omega & \text{if } \ell(s\omega) < \ell(\omega) \end{cases} \tag{1.2}$$

Skip these proofs if they are all in the book! R.

Theorem 1.2.3. *$\mathscr{H}_{R,q}$ is a free R-algebra with basis $\{T_\omega \mid \omega \in S_n\}$*

Remark 1.2.4. *If R is a field then $\mathscr{H}_{R,q}$ is isomorphic to a matrix algebra in $M_n(R)$.*

This process is called the specialisation. As.

Corollary 1.2.5. *$\phi : \tilde{R} \to R$ is a ring homomorphism with $\phi(\tilde{q}) = q$ for some $\tilde{q} \in R$ then*

$$\mathscr{H}_{R,q} \cong \mathscr{H}_{\tilde{R},q} \otimes_{\tilde{R}} R$$

Definition 1.2.6. *There is a bilinear form $(\cdot,\cdot) : \mathscr{H} \times \mathscr{H} \to R$ given on h_1, h_2 by the coefficient in R of the unit 1 in $h_1 \cdot h_2$.*

You know what else got a nice bilinear form? Cellular algebras! Is is a coincidence that the next chapter is on them? I think not! Al.

Proposition 1.2.7. *Let $v, \omega \in S_n$.*

$$(t_v, t_\omega) = \begin{cases} q^{\ell(v)} & v = \omega^{-1}; \\ 0. \end{cases} \tag{1.3}$$

So, the bilinear form is symmetric and associative. If q is invertible, then it is also non-degenerate.

As R is a domain, should it matter that q is invertible or not for non-degeneraty? Si., R.
Reading the book, I think he got a bit confused because in the introduction he does not ask for R to be a domain. Al.

Definition 1.2.8. *A is an algebra over field R. $e \in A$. is idempotent $e^2 = e$. Then $H(A,e) := eAe$. is the Hecke algebra corresponding to (A,e).*

Prove the following statement

1. $H(A,e) \cong \operatorname{End}_A(Ae)$

2. A semisimple $\Rightarrow H(A, e)$ semisimple one-to-one correspondence.

$$\{\text{irr comp of } Ae\} \leftrightarrow \{\text{irr rep of } H(A, e)\}$$

3. Dimension of $H(A, e)$ irreducible representation is the multiplicity of corresponding irreducible components of Ae.

This is exercice 12 of chapter 1

Chapter 2

Going all cellular

Presented by Alexis Langlois-Rémillard on 04-10-2019, 08-10-2019 and 22-10-2019.
Notes recorded by Wouter van de Vijverand Alexis Langlois-Rémillard.

2.1 First presentation

Given 04-10-2019

In this section we present the definition of Graham and Lehrer. After this definition is recalled we introduced some basic consequences and reach the statement of the first useful theorem.

Mathas use this ordering that is the inverse of the original one by Graham and Lehrer. It has become more and more standard to do so [2]. Al.

Definition 2.1.1 (Graham and Lehrer [4]). *Let R be a commutative associative unitary ring. An R-algebra A is called* cellular *if it admits a cellular datum $(\Lambda, T, C, *)$ consisting of the following:*

1. *a finite partially-ordered set Λ and, for each $\lambda \in \Lambda$, a finite set $T(\lambda)$;*
2. *an A-basis $\{c^{\lambda}_{s,t} \mid \lambda \in \Lambda, s, t \in T(\lambda)\}$*
3. *an anti-involution $* : A \to A$ such that*
$$(c^{\lambda}_{s,t})^* = c^{\lambda}_{t,s} \qquad \text{for all } s, t \in T(\lambda); \tag{2.1}$$
4. *if $\lambda \in$ Lambda and $s, t \in T(\lambda)$, then for any $a \in A$,*
$$c^{\lambda}_{s,t} a \equiv \sum_{s' \in T(\lambda)} r_a(t', t) c^{\lambda}_{s,t'} \ \ mod\ A^{>\lambda}, \tag{2.2}$$
where $A^{>\lambda} = \left\langle c^{\mu}_{p,q} \mid \mu > \lambda; p, q \in T(\mu) \right\rangle_R$ and $r_a(s', s) \in R$ is independent of t.

In Graham and Lehrer, it reads "an injective map $C : \bigsqcup_{\lambda\in\Lambda} T(\lambda)\times T(\lambda) \to A$ whose image is an R-basis of A, with the notation $C^{\lambda}(s, t)$ for the image under C of the pair $(s, t) \in T(\lambda) \times T(\lambda)$;"

The involution $*$, together with (2.2), yields the equation:

$$a^* c^\lambda(t,s) \equiv \sum_{t' \in T(\lambda)} r_a(t',t) c^\lambda_{t',s} \;\; mod \; A^{>\lambda}, \tag{2.3}$$

for all $s, t \in T(\lambda)$ and $a \in A$.

Examples Now time for examples! First is the algebra of polynomial over a field $\mathbf{k}[x]$ with anti-involution $x \mapsto -x$, poset $\mathbb{N}$, sets $T(n) = \{n\}$ and cellular basis $\{x^n \mid n \in \mathbb{N}$.

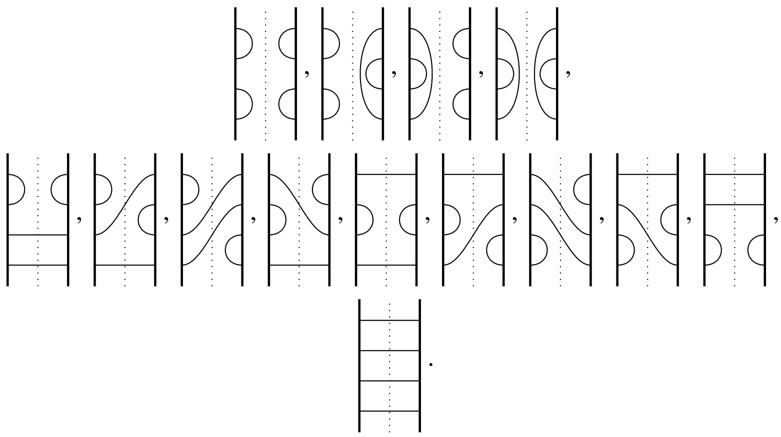
This comes from the f Alexis and Yvan Saint-Aubin section 3.1 with a slight change for the poset as they took the definition of Graham and Lehrer.

Second are the Temperley-Lieb algebras. For a certain integer n and an arbitrary invertible complex number q, the algebra is the Temperley-Lieb algebra introduced by Temperley and Lieb. We use the graphical notation of Kauffmann. In this setting, the Temperley-Lieb algebra $\mathsf{TL}_n(q+q^{-1})$ is the $\mathbb{C}$-algebra generated by formal sums of (n,n)-diagrams with multiplication being concatenation and resolution of closed loops by multiplication by $q+q^{-1}$. A (n,n)-diagram is build by placing two sets of n dots on two parallel lines and linking the $2n$ dots by non-intersecting line in the rectangle made by the two lines.

To show that the Temperley-Lieb algebras are cellular, we will give the cellular datum and show it for TL_4.

You can have the dimension of the Temperley-Lieb algebras via this argument, it is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The poset is the set of possible number of arc linking two points on the same line of a diagram. It is $\{0, 1, \ldots, \lfloor n/2 \rfloor\}$. For each λ in the poset, the set $T(\lambda)$ is given by half-diagram with λ arcs. The basis is given by the (n,n) diagram. The involution is given by vertical reflection. The last axiom is respected because it is impossible to destroy arcs (and thus one can only go higher on the number of arc created).



The third example is given in Mathas and is the Hecke-algebra.

Before continuing, let us define an important family of modules coming directly from both definitions.

Definition 2.1.2. *Let A be a cellular algebra with cellular datum $(\Lambda, T, C, *)$. Let $\lambda \in \Lambda$ and consider C^λ as the free R-modules with basis $\{v^\lambda_s \mid s \in T(\lambda)\}$. When endowed with the (right) A-action defined by*

$$v^\lambda_s := \sum_{s' \in T(\lambda)} r_a(s',s) v^\lambda_{s'} \quad \text{for } a \in A, \tag{2.4}$$

the (right) A-module C^λ is called a cell module *of A.*

The coefficients $r_a(t', t)$ defined through axiom (2.2) are used to construct cellular modules, but they are even richer. If p, s, t and $u \in T_A(\lambda)$ for some $\lambda \in \Lambda_A$, then equations (2.2) and (2.3) lead to two distinct expressions for the product $c^\lambda_{p,s} c^\lambda_{t,u}$:

$$\sum_{t'} r_{c^\lambda_{p,s}}(t', t) c^\lambda_{t',u} \equiv \sum_{s'} r_{c^\lambda_{u,t}}(s', s) c^\lambda_{p,s'} \ \ mod\ A^{>\lambda}. \tag{2.5}$$

Since the c's form a basis of A, only the term $t' = p$ in the left hand side and the term $s' = u$ in the right hand side may contribute in each sum. Thus

I hope you will forgive my slight abuse of notation with the r's (putting r(u, v)! Al.

$$r_{c^\lambda_{p,s}}(p, t) = r_{c^\lambda_{u,t}}(u, s).$$

Since the left member is independent of u and the right one of p, it follows that both of these coefficients depend only on s and t. This fact is emphasized by writing

$$c^\lambda_{p,s} c^\lambda_{t,u} \equiv r^\lambda(s, t) c^\lambda_{p,u} \ \ mod\ A^{>\lambda},$$

with $r^\lambda(s, t) := r_{c^\lambda_{p,s}}(p, t) = r_{c^\lambda_{u,t}}(u, s)$.

Definition 2.1.3. *A bilinear form $\langle -, - \rangle^\lambda : \mathsf{C}^\lambda \times \mathsf{C}^\lambda \to R$ on the cellular module C^λ is defined by $\langle v_s, v_t \rangle = r^\lambda(s, t)$.*

This bilinear form plays a central role in the theory of cellular algebra because of the following result.

Proposition 2.1.4 (Graham and Lehrer, Prop. 2.4, [4])**.** *The bilinear form $\langle -, - \rangle^\lambda$ on C^λ, $\lambda \in \Lambda$, has the following properties.*

1. *It is symmetric: $\langle x, y \rangle^\lambda = \langle y, x \rangle^\lambda$ for all $x, y \in \mathsf{C}^\lambda$.*
2. *It is invariant: $\langle a^* x, y \rangle^\lambda = \langle x, ay \rangle^\lambda$ for all $x, y \in \mathsf{C}^\lambda$ and $a \in A$.*
3. *If $x \in \mathsf{C}^\lambda$ and $s, t \in T(\lambda)$, then $c^\lambda_{s,t} x = \langle v_t, x \rangle^\lambda v_s$.*

We define for any cellular algebra Λ_0 to be the subset of Λ in which the bilinear form just defined is not identically zero. The *radical* of the bilinear form $\langle -, - \rangle^\lambda$ is denoted R^λ. As the form is invariant, it is a submodule of C^λ because of the invariance of the form. However, there is even more to it.

The radical of a bilinear form is the set of element x such that $\langle x, y \rangle = 0$ for all y.

Proposition 2.1.5. *Let A be a cellular algebra over a field $\mathbf{k}$ and $\lambda \in \Lambda_0$. The radical R^λ of the bilinear form $\langle -, - \rangle^\lambda$ is the Jacobson radical of C^λ, and the quotient $\mathsf{D}^\lambda := \mathsf{C}^\lambda / \mathsf{R}^\lambda$ is absolutely irreducible.*

Proof. We take an x outside the radical and find another element y in the module such that their bilinear form gives 1. It is possible because the algebra is over a field and the bilinear form is not totally zero. Then, for any element v we want, we can create an element of the algebra y_v coming from y such that $x y_v = v$ by using the properties of the bilinear form. This proves that the radical of the bilinear form is the only maximal submodule and thus that it is the Jacobson radical. □

Note that this proof says that the cellular modules are cyclic when $\lambda \in \Lambda_0$ over a field.

We now have some technical lemmas.

Lemma 2.1.6. *If $a \in \mathsf{C}^\lambda$ and $y \in A^{\geq \mu}$ then $ay = 0$ unless $ay = 0$, that is, if $\lambda \ngeq \mu$, then $ay = 0$.*

The following proposition gives conditions for morphism.

Proposition 2.1.7. *Let $\lambda \in \Lambda$ and $\mu \in \Lambda_0$. Let M be a proper submodule of C^λ and suppose that $\theta : \mathsf{C}^\mu \to \mathsf{C}^\lambda/M$ is a morphism of A-modules.*

1. *If $\theta \neq 0$ then $\lambda \geq \mu$.*
2. *If $\mu = \lambda$ then there exists a $r_\theta \in R$ such that $\theta(z) = M + r_\theta z$ and thus $Hom(\mathsf{C}^\mu, \mathsf{C}^\lambda/M) \simeq \mathbf{k}$.*

We finished here with the statement of the theorem 2.2.4 that we moved further for these notes.

Corollary 2.1.8. *If $\mathsf{D}^\mu \simeq \mathsf{D}^\lambda$ are not zero, then $\lambda = \mu$.*

2.2 Second presentation

On 08-10-2019.

For any finite poset, you can have a total ordering so this is a very natural lemma indeed. As.

Definition 2.2.1. *A subset Γ of Λ is called a poset ideal of Λ if for every $\lambda \in \Gamma$, anytime a $\mu \in \Lambda$ is such that $\mu > \lambda$, then $\mu \in \Gamma$.*

We write $A(\Gamma)$ for the $\mathbf{R}$-module of A with basis $\{c^\mu_{u,v} \mid \mu \in \Gamma, u, v \in T(\mu)\}$. Then $A(\Gamma) = \sum_{\mu\in\Gamma} A^{\leq\mu}$ is a two-sided ideal of A.

With this we have a lemma giving a filtration of A. If elsewhere it was sometime possible to remove the condition of finiteness for Λ, there it is important, or at least it is required to have a total ordering .

The proof take a total ordering and then yields immediatly a filtration and maximality assures us that the proposed definition of Γ works and that the quotient will be generated by elements of the form $c^{\mu_i}_{u,v} + A^{>\mu_i}$. Al.

Lemma 2.2.2. *Let $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Lambda$ be a maximal chain of poset ideals in Λ. There exist a total ordering $\mu_1, \ldots, \mu_k$ of Λ such that $\Gamma_i = \{\mu_1, \ldots, \mu_i\}$ for all i and*

$$0 = A(\Gamma_0) \hookrightarrow A(\Gamma_1) \hookrightarrow \ldots \hookrightarrow A(\Gamma_k) = A \tag{2.6}$$

is a filtration with composition factor $A(\Gamma_i)/A(\Gamma_{i-1}) \simeq (\mathsf{C}^{\mu_i})^* \otimes_R \mathsf{C}^{\mu_i}$

Remark that this lemma means that A has a filtration that can be extended to a composition series made only with cellular modules and their composition factors.

We need another lemma before proving the theorem. It is a rather intuitive statement saying that for minimal elements, the cellular modules are simple.

Lemma 2.2.3. *If λ is a minimal element of Λ then $\mathsf{C}^\lambda \simeq \mathsf{D}^\lambda$.*

The real reason this is here is because Alexis messed up the indices in the talk, but I forgive him. Al.

Now we enter the main theorem that will be proved here in somewhat details.

Theorem 2.2.4. *The set $\{\mathsf{D}^\lambda \mid \lambda \in \Delta^0\}$ is a complete set of non-isomorphic (absolutely) irreducible modules.*

If the distinction between composition series and filtration is not clear, it might be a good idea to review the appendix A of Mathas, or chapter VIII of Curtis and Reiner [1].

Proof. is already known that all of the module D^λ are simple by the proposition giving the equivalence between the radical of the bilinear form and the Jacobson radical, and no two are isomorphic by the corollary of this proposition. Thus, it is required to prove that any simple module D is in fact of the form D^λ.

The lemma 2.2.2 gives a filtration of A by cellular modules quotient and thus it will be sufficient to prove that any composition factor of a cellular module is of the form D^λ to prove this. The proof proceeds by induction.

The base case is given by a minimal λ in which case the preceding lemma assures that the module C^λ is simple and thus it has only one composition factor.

We proceed by induction on this λ. Assume as inductive hypothesis that all the composition factors of the modules C^μ for $\mu < \lambda$ are of the form D^ν for a certain ν.

Let D be a (simple) composition factor of C^λ. We know that C^λ has the radical filtration given by $\emptyset \subset \mathsf{R}^\lambda \subset \mathsf{C}^\lambda$. Thus D is either $\mathsf{D}^\lambda = \mathsf{C}^\lambda/\mathsf{R}^\lambda$ or a composition factor of the radical of C^λ. Consider D as a composition factor of R^λ.

Consider the complement to the set of inductive hypothesis $\Gamma = \{\eta \in \Lambda \mid \lambda \not> \eta\}$; it is a poset ideal and thus $A(\Gamma)$ is an ideal of A. For any element of R^λ is annihilated by $A^{\geq\lambda}$ by properties of the bilinear form: indeed without loss of generality let $x \in \mathsf{R}^\lambda$ and $c_{s,t} \in A^{\geq\lambda}$, then

$$xc_{s,t} = \langle x, v_s \rangle v_t = 0. \tag{2.7}$$

Samewise, we have $\mathsf{C}^\lambda A^\eta$ when $\eta \in \Gamma$ is such that $\gamma \neq \lambda$ by lemma 2.1.6. Combining the two statement we have

$$\mathsf{R}^\lambda A(\Gamma) = 0$$

and thus every composition factor of R^λ is a composition factor of $A/A(\Gamma)$. The two filtrations $\emptyset \subset A(\Gamma) \subset A$ and $\emptyset \subset \mathsf{R}^\lambda \subset A$ are equivalent.

Now extend the filtration $\emptyset \subset \Gamma \subset \Lambda$ to a maximal chain of poset ideals and use lemma 2.2.2 to get a filtration for $A/A(\Gamma)$ with all their composition factors cellular modules C^ν with $\nu \notin \Gamma$ and thus $\mu < \lambda$ and by induction hypothesis we have that all the composition factors of C^μ are isomorphic to some D^ν.

□

The rest is an introduction with some of the vocabulary used in the main theorem.

The decomposition multiplicity is the number of time there are simple quotient isomorphic to a certain simple module in the composition series.

Definition 2.2.5. *The decomposition matrix* $\mathbf{D}$ *of* A *is the* $|\Lambda| \times |\Lambda_0|$ *matrix defined by the composition multiplicities of the simple modules in the cellular modules,*

$$\mathbf{D} = ([\mathsf{C}^\lambda : \mathsf{D}^\mu])_{\substack{\lambda\in\Lambda \\ \mu\in\Lambda_0}}. \tag{2.8}$$

A corollary of the proposition 2.1.7 give the peculiar form of $\mathbf{D}$.

Corollary 2.2.6. *The matrix* $\mathbf{D}$ *is a unitriangular superior matrix.*

For $\lambda \in \Lambda_0$ we have a unique D^λ. By correspondance between simple and principal modules, there is a unique indecomposable projective module P^λ characterized by $\mathsf{P}^\lambda / \mathit{rad}\ \mathsf{P}^\lambda \simeq \mathsf{D}^\lambda$.

We will use principal module or projective indecomposable indistinctively. They are the indecomposable blocks of the regular representation A_A.

There is already a link between the principal module and the entries of $\mathbf{D}$.

Lemma 2.2.7. *Let* $\lambda \in \Lambda$ *and* $\nu \in \Lambda_0$.

$$[\mathsf{C}^\lambda : \mathsf{D}^\nu] = dim_R\, Hom_A(\mathsf{P}^\nu, \mathsf{C}^\lambda) = dim\, \mathsf{P}^\nu \otimes_R (\mathsf{C}^\lambda)^* \tag{2.9}$$

Definition 2.2.8. *The Cartan matrix* $\mathbf{C}$ *of* A *is the square matrix of dimension* $|\Lambda_0|$ *of entries the composition multiplicities of the simple in the principal modules.*

$$\mathbf{C} = ([\mathsf{P}^\lambda : \mathsf{D}^\mu])_{\lambda,\mu\in\Lambda_0}. \tag{2.10}$$

The main theorem is that $\mathbf{C} = \mathbf{D}^t\mathbf{D}$. Next presentation will prove this.

2.3 Third presentation

On 22-10-2019.

In this presentation we present the last important lemma and the main theorem of the theory of cellular algebras.

Note that with modules, the tensor product might reduce the dimension! So it won't blow up

Lemma 2.3.1. *Let P be a projective A-module and note $|\Lambda| = k$. Then P admits an A-modules filtration*

$$\emptyset = P_0 \subset P_1 \subset \cdots \subset P_k = P \tag{2.11}$$

such that the non-zero factor modules $P_i/P_{i=1}$ are isomorphic to the (non-zero) modules $P \otimes_A ((\mathsf{C}^\nu)^* \otimes_R \mathsf{C}^\nu)$ with each $\nu \in \Lambda$ occuring exactly once.

The main theorem of the theory links the decomposition factors of the indecomposable projective modules and the irreducible ones. Denote by P^d the projective cover of D^d and let $\mathbf{D} = ([\mathsf{C}^d : \mathsf{D}^e])_{d\in\Delta, e\in\Delta^0}$ be the decomposition matrix of A and $\mathbf{C} = ([\mathsf{P}^d : \mathsf{D}^e])_{d,e\in\Delta^0}$, the Cartan matrix.

Theorem 2.3.2 (Graham and Lehrer, Thm. 3.7 [4])**.** *The matrices* $\mathbf{C}$ *and* $\mathbf{D}$ *are related by*

$$\mathbf{C} = \mathbf{D}^t\mathbf{D}. \tag{2.12}$$

Chapter 3

Modular representation theory of Hecke algebras

Presented by Sigiswald Barbier, Wouter van de Vijver and Sam Claerebout on DATE.
Notes recorded by X and Y.

3.1 Combinatorics of Young tableaux

Presented by Sigiswald Barbier on DATE.
Notes recorded by X and Y.

3.2 Murphy basis

Presented by Sigiswald Barbier on DATE.
Notes recorded by X and Y.

3.3 Irreducible representation

Presented by Wouter van de Vijver on DATE.
Notes recorded by X and Y.

3.4 Kazhdan-Lutszig polynomials

Presented by Sam Claerebout on DATE.
Notes recorded by X and Y.

Chapter 4

Double Affine Hecke Algebras

Presented by Roy Oste and Hadewijch De Clercq on DATE.
Notes recorded by X and Y.

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