Categorification

by examples

Seminar transcript from
the Winter 2020 Kleine Seminar

Sigiswald Barbier
Asmus Bisbo
Sam Claerebout
Ali Guzmán Adán
Paulien Jansen
Alexis Langlois-Rémillard
Roy Oste
Wouter van de Vijver
Kleine Seminar, Universiteit Gent

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Chapter 1

Introduction

Welcome to the *Kleine Seminar*! Introduced in the summer 2019 by a small gathering of (post)-doctoral students at Universiteit Gent, the Kleine Seminar is thought to be held during the course session and has the objective to study some subjects the members are interested in. For each subject, every member will get to present a piece of the material.

Subjects are chosen on a unanimous vote from the presenting members

This small book groups the minutes of the meetings based upon a member live-TEXing and the presenter own notes. Some comments in the margin are placed through the text to represent the discussion, formal and informal, the members had. It is based loosely in the graffiti the interesting Concrete Mathematics book by Graham and Knuth [4]. Some of them are signed if the writer remembers (and deem the correct worthy of authorship).

It's not really mandatory, but it does add somewhat a more discussing tone!

The Winter 2020 session aims to introduce the topics of categorification by working out examples. It is a relatively new topic that gained a lot of attention from physics and mathematics alike. The realistic goal of the seminar is to be able to understand better its vocabulary to follow the advances made and hopefully be able to use some of the insights.

All comments are welcomed, corrections can be directly made by one of the member, so contacting any of them should work, as long as they are still at Universiteit Gent.

Sincerely, the committee

Sigiswald Barbier;
Asmus Bisbo;
Sam Claerebout;
Ali Guzmán Adán;
Paulien Jansen;
Alexis Langlois-Rémillard;
Roy Oste;
Wouter van de Vijver.

Chapter 2

Some categorical prerequisites and first string diagrams

Presented by Wouter van de Vijver on 12-00-2020.
Notes recorded by Alexis Langlois-Rémillard.

The references for this course will be

- *String diagrams and categorification* from Alistair Savage [5]
- *Linear algebraic groups* from Tom De Medts [1]
- *Introduction à la théorie des schémas* from M. Ducros [3].

2.1 Categories, a dictionary

The vocabulary of categories is briefly recalled.

2.1.1 Categories

We call C a category if it has

W: The use of class is to avoid the logical minefield; when the morphism sets are sets, then it is a (locally) small categories

- A class: $\mathrm{OB}(\mathsf{C})$
- A class $\mathrm{HOM}(\mathsf{C})$ of arrows $\alpha : X \to Y$ for any two objects X and Y, X such that there is always a map $id_X : X \to X$ and there is a composition law from $Hom(X,Y) \times Hom(Y,Z) \to Hom(X,Z)$ any three objects X,Y,Z given by $(\alpha,\beta) \to \beta \circ \alpha)$. The identity map is the identity for the monoid Hom

Examples:

1. For any monoid M, there is a category of one object X with $Hom(X) = Hom(X, X) = M$. (It is a small category)

2. The category Set of sets, with sets as objects and maps as arrows

3. The category Grp with objects groups and arrows homomorphisms of groups

4. The category Top with objects topological spaces and arrows continuous maps

Let C be a category and S an object of C. There is a category $S\backslash\mathsf{C}$ with object given by (X, f) with X an object of C, $S \xrightarrow{f} X$ and arrows $Hom(S\backslash C)$ given by $\alpha \in Hom((X, f), (Y, g))$ it commutes $\alpha \circ f = g$, which in terms of commutative diagrams gives:

$$\begin{tikzcd} X \arrow[rr, "\alpha"] & & Y \\ & S \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd} . \tag{2.1}$$

Examples: The left modules $_RMod$ and right ones Mod_R are examples of such construction.

As.: To get this, it is not crystalline. You'll have to take $\mathsf{C} = \mathsf{Ab}$ *and the object* R *to be a ring, then you will get* R*-module.*

Another important example is the *opposite category* given by reversing all the arrows.

We can define a lot of "usual" things with category. Properties of arrows are related to properties of morphism.

- $f : X \to Y$ is an epimorphism if for $g_1, g_2 : Y \to Z$ the following $g_1 \circ f = g_2 \circ f$ implies \$ $g1 = g2$. (Make diagram with double arrow)

- $h : Z \to X$ is a monomorphism if for $g_1, g_2 : Y \to Z$ the following $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

- It is an isomorphism if it is has an inverse.

In specialising in categories, we have back some of the usual construction.

2.1.2 Functors

Let C, D be two categories. We call $F : \mathsf{C} \to \mathsf{D}$ a functor if for every object $X \in \mathsf{C}$, $F(X)$ is an object of D; F sends morphism of C to morphism of D in either

- $F(f : X \to Y) \in Hom(F(X), F(Y))$ if F is *covariant*

- $F(f : X \to Y) \in Hom(F(Y), F(X))$ if F is *contravariant*

and the functor sends the identity to the identity and it respects the composition law.

Examples Forgetful functors: a functor that "forget" some of the structure. For examples, you can go from Grp to Set with the covariant functor that forget the group action. Samely, you can go from Vect to Ab.

There is a (contravariant) functor from Top to k-algebra $_kAlg$ sending $M \to \{f : M \to k\}$. For N, M two topological spaces and $\alpha : N \to M$, the functor goes from $F(M)$ to $F(N)$ sending f to $f \circ \alpha$.

Let C be locally small category. There is thus two functors

- $Hom(X, -) = h_X : \mathsf{C} \to \mathsf{Set}$ sending $Y \mapsto Hom(X, Y)$ (covariant);
- $Hom(-, Y) = h^Y : \mathsf{C} \to \mathsf{Set}$ sending $X \mapsto Hom(X, Y)$ (contravariant).

A functor is *full* if $f : F(f)$ in $Hom(X, Y) \to Hom(F(X), F(Y))$ is surjective. It is *faithful* if this map is injective. It is *fully faithful* if it is both

2.1.3 Natural transformations

We say C and D are isomorphic if there are functors $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{C}$ such that $G \circ F = id_{\mathsf{C}}$ and $F \circ G = id_{\mathsf{D}}$.

W:Sometime called morphism

Let's define another concept, *natural transformation*. It is a kind of morphism of functors. $\phi : F \to G$ is a natural transformation if for $X \in \mathsf{C}$ then $\phi(X) : F(X) \to G(X)$. and for $\alpha : X \to Y$ the diagram commutes ($\phi(Y) \circ F(\alpha) = G(\alpha) \circ \phi(X)$).

W: This concept is too strong, we want more flexibility.

PROBLEM

We say $F \simeq G$ if there are two natural transformations $\phi : F \to G$ and $\psi : G \to F$ such that $\phi \circ \psi = id_F$ and $\psi \circ \phi = id_G$.

SOLUTION:

MacLane: it is isomorphic if every $\phi(X)$ is invertible.

We say C, D to be equivalent if there is $F : \mathsf{C} \to \mathsf{D}, G : \mathsf{D} \to \mathsf{C}$ such that $G \circ F = id_{\mathsf{C}}$ and $F \circ G = id_{\mathsf{D}}$.

Examples For $M \in {_A}Mod$ there is a natural transformation sending M to its bidual $M \to M^{vv}$ sending $v \to (\phi \to \phi(v))$

2.2 Strict monoidal category

We take C a locally small category boasting

1. a bifunctor: $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ called the *tensor product*;
2. a unit object 1;

such that for every (X, Y, Z) the tensor product is associative both on objects ($X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$) and morphisms ($(f \otimes g) \otimes h = f \otimes (g \otimes h)$) and $1 \otimes X = X = X \otimes 1$ and $id_1 \otimes f = f = f \otimes id_1$.

It is called strict because the above are equality. MacLane proved in his Categories for the working mathematicians that it does not really matter if you take non-strict monoidal categories are there are always a strict one isomorphic to it.

Linear category

k is a commutative ring.

A category C is k-linear if for every objects X, Y, $Hom(X, Y)$ is a k-module and the operation respect the composition:

$$(\alpha_1 f_2 + \alpha_2 f_2) \circ g = \alpha f_1 \circ g + \alpha_2 f_2 \circ g \tag{2.2}$$
$$f \circ (\beta_1 g_1 + \beta_2 g_2) = \beta_1 f \circ g_1 + \beta_2 f \circ g_2. \tag{2.3}$$

Being a bifunctor for the tensor product means that there is an interchange law going on:

$$(1_X \otimes g) \circ (f \otimes 1_Y) = ... \tag{2.4}$$
$$= (f \otimes 1_Y) \circ (1_X \otimes g) \tag{2.5}$$

Examples

If we take the (strictly monoidal) category of one object, the morphism set $End(1)$. Then we have

$$f \circ g = (1 \otimes f) \circ (g \otimes id) = g \otimes f = (g \otimes 1) \circ (1 \otimes f) = g \circ f$$

and thus, it is the center of C, a commutative monoid.

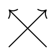
2.3 String diagrams

This is an example extracted from Alistair Savage's note [5]:

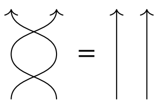
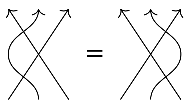
2.3.1 The symmetric group

As a concrete example, define $\mathcal{S}$ to be the strict k-linear monoidal category with:

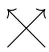
- one generating object $\uparrow$,
- one generating morphism

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow,$

- two relations

 and  . (2.6)

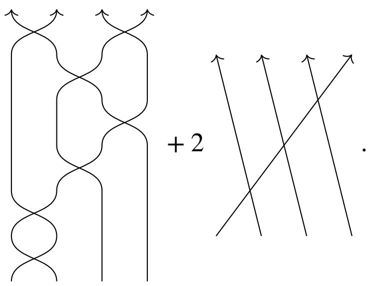
One could write these relations in a more traditional algebraic manner, if so desired. For example, if we let

$s =$  $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow,$

then the two relations Equation (2.6) become

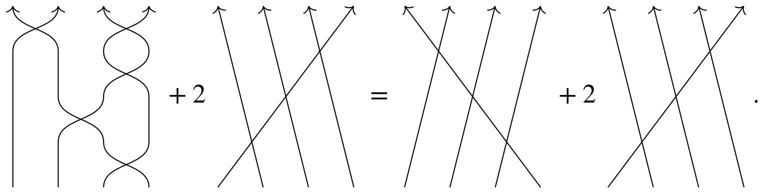
$$s^2 = 1_{\uparrow\otimes\uparrow} \quad \text{and} \quad (s \otimes 1_\uparrow) \circ (1_\uparrow \otimes s) \circ (s \otimes 1_\uparrow) = (1_\uparrow \otimes s) \circ (s \otimes 1_\uparrow) \circ (1_\uparrow \otimes s).$$

An example of an endomorphism of $\uparrow^{\otimes 4}$ is



Using the relations, we see that this morphism is equal to

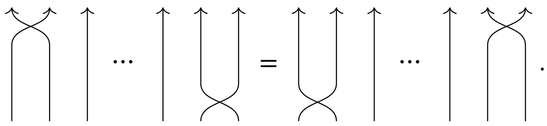
W: very nice, way less relations needed!



Fix a positive integer n and recall that the group algebra $k S_n$ of the symmetric group on n letters has a presentation with generators $s_1, s_2, \dots, s_{n-1}$ (the simple transpositions) and relations

$$s_i^2 = 1, \qquad 1 \le i \le n-1, \tag{2.7}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad 1 \le i \le n-2, \tag{2.8}$$
$$s_i s_j = s_j s_i, \qquad 1 \le i, j \le n-1,\ |i-j| > 1. \tag{2.9}$$

Commutativity of distant element is given for free by monoidal.



Chapter 3

Pivotal categories

Presented by Asmus Bisbo on 19-02-2020. Note recorded by Alexis Langlois-Rémillard

3.1 Definition of adjunction and other preliminaries

Definition Let C and D be two locally small categories with functors $F : D \to C$ and $G : C \to D$ such that for each $X \in C$ and $Y \in D$ there is a map $\phi_{XY} : Hom_C(FY, X) \to Hom_D(Y, GX)$. If the map ϕ_{XY} is a bijective natural transformation in C and D then F is called *left adjoint* of G and G is called *right adjoint* of F.

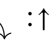
Example

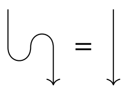
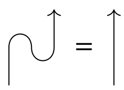
It looks like dual. For V, W finite dimensional vector space with an inner product and $T : V \to W$ a linear transform. Then the dual $T^* : W \to V$ with respect to the inner product is such that $(Tv, w) = (v, T^*w)$.

Definition (counit-unit)

Let C and D be two locally small categories, $F : \mathsf{D} \to \mathsf{C}$ and $G : \mathsf{C} \to \mathsf{D}$ and natural transformations $\varepsilon : FG \to 1_{\mathsf{C}}$ (counit) and $\eta : 1_{\mathsf{D}} \to GF$ (unit). If $\varepsilon F \circ F\eta = 1_F$ and $G\varepsilon \circ \eta G = 1_G$ then F is left adjoint of G and G is right adjoint of F.

[Discussion with Sigiswald and Alexis because it is slightly unclear]

Consider strict monoidal category with objects ↑ and ↓ and morphisms: and morphism  : $1 \to \downarrow \otimes \uparrow$ and  $: \uparrow \otimes \downarrow \to 1$. We call ↓ the right dual of ↑ and ↑ the left dual of ↓ if the following holds:

 and  . (3.1)

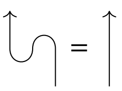
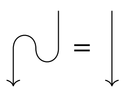
It goes then to in

$(1_\downarrow \otimes$  $) \circ ($  $\otimes 1_\downarrow) = 1_\downarrow$ and ( $\otimes 1_\uparrow) \circ 1_\uparrow \otimes$  $) = 1_\uparrow$.

If they are both right and left dual there is also the same way in the other direction

 $: 1 \to \uparrow \otimes \downarrow$ and  $: \downarrow \otimes \uparrow \to 1$

such that

 and  . (3.2)

Definition
If all object in a strict monoidal category C have right and left dual, C is called *rigid* or *autonomous*.

Let us go to vector space example Let $\mathbf{k}$ be a field and $V \in \mathsf{Vect}_{\mathbf{k}}$. The dual V^* of V is also an object of $\mathsf{Vect}_{\mathbf{k}}$. Fix a basis B of V and a dual basis $\{\delta_v : v \in B\}$ of V^*. Doing the identification $V \to \uparrow$ and $V^* \to \downarrow$; $\mathbf{k} \to 1$, we then have the map

$$\cup : \mathbf{k} \to V^* \otimes V \qquad \cap : V \otimes V^* \to \mathbf{k} \tag{3.3}$$

$$\alpha \mapsto \alpha \sum_{v \in B} \delta_v \otimes v \qquad \sum_{i=1}^{n} v_i \otimes f_i \mapsto \sum_{i=1}^{n} f_i(v_i) \tag{3.4}$$

$$\cup : \mathbf{k} \to V \otimes V^* \qquad \cap : V^* \otimes V \to \mathbf{k} \tag{3.5}$$

$$\alpha \mapsto \alpha \sum_{v \in B} v \otimes \delta_v \qquad \sum_{i=1}^{n} f_i \otimes v_i \mapsto \sum_{i=1}^{n} f_i(v_i). \tag{3.6}$$

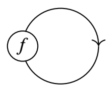
Writing

$$V \xrightarrow{\sim} V \otimes \mathbf{k} \xrightarrow{1_V \otimes \cup} V \otimes V^* \otimes V \xrightarrow{\cap \otimes 1_V} \mathbf{k} \otimes V \xrightarrow{\sim} V$$

For $w \in V$ this gives

$$w \mapsto w \otimes 1 \mapsto \sum_{v \in B} w \otimes \delta_v \otimes v \mapsto \sum_{v \in B} \delta_v(w) \otimes v \mapsto \sum_{v \in B} \delta_v(w) v = w$$

Trace
We define an operator in $f \in End(\uparrow)$ that get represented as



$$, \quad f \in End(\uparrow). \tag{3.7}$$

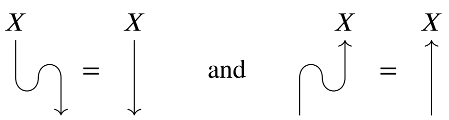
Example
the circle corresponds to

$$\mathbf{k} \to V \otimes V^* \xrightarrow{f \otimes 1_{V^*}} V \otimes V^* \to \mathbf{k}$$

then

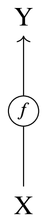
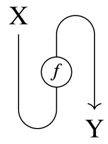
$$\alpha \mapsto \alpha \sum_{v \in B} v \otimes \delta_v \mapsto \sum_{v \in B} f(v) \otimes \delta_v \mapsto \alpha \sum_{v \in B} \delta_v(f(v)) = \alpha tr(f)$$

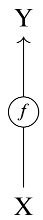
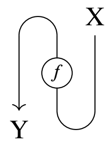
Definition
Let C be a strict monoidal category and X an object of C with its right dual X^*. Then we write $1_X = \uparrow_X$ and $1_{X^*} = \downarrow_X$ with morphism ${}^X\!\cup : 1 \to X^* \otimes X$ and ${}_X\!\cap : X \otimes X^* \to 1$. Such that



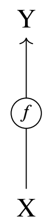
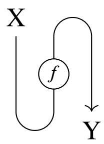
$$. \tag{3.8}$$

Suppose X, Y are objects of C with respective right dual X^*, Y^*. Then for any $f \in Hom(X, Y)$, the string diagram with decoration of f then gets

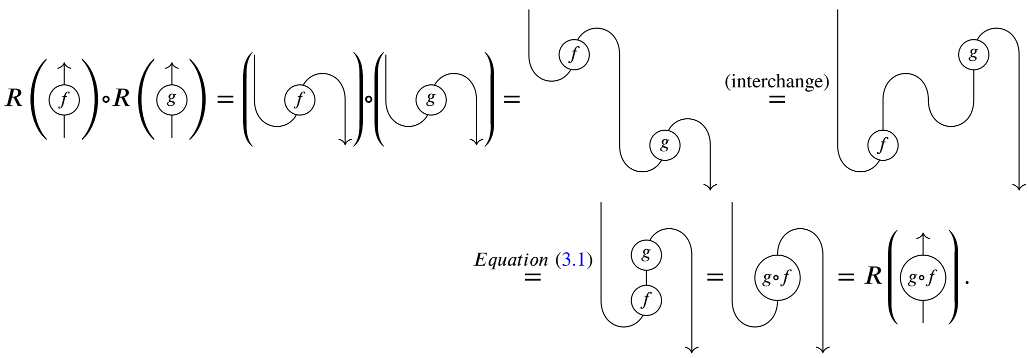
 $\in End(X, Y)$ has right mate  $\in End(Y^*, X^*)$ (3.9)

 $\in End(X, Y)$ has left mate  $\in End(Y^*, X^*)$ (3.10)

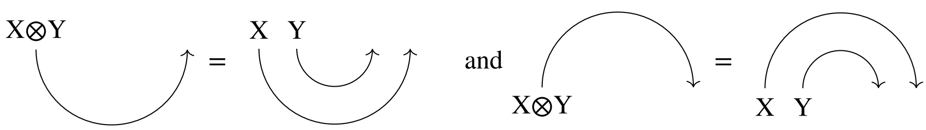
Let us consider the map $R : \mathsf{C} \to \mathsf{C}$ that takes f to its right mate

$R :$  $\longmapsto$  . (3.11)

Then as we want to see if it becomes a functor. Turns out, it is a contravariant. We extract the diagrammatic proof of Alistair Savage [5]



We introduce a notation for tensoring cups and caps. Let X, Y be two objects of C.



Pivotal category A strict monoidal category C is called strict pivotal if X has right dual X^* for all object X such that $(X^*)^* = X$ for all object in C. (Thus R is an "anti-involutive endofunctor").

Examples Temperley-Lieb

The Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$ has $n-1$ generators of $U_1, \dots, U_{n-1}$ and element β satisfying relations

$$U_i^2 = \beta U_i \tag{3.12}$$
$$U_i U_{i\pm 1} U_i = U_i \tag{3.13}$$
$$U_i U_j = U_j U_i \quad (|i-j| > 1) \tag{3.14}$$

I recommend the article by Ridout and Saint-Aubin [2] as an introduction to Temperley-Lieb algebras, albeit very much without the mention of category!

The Temperley-Lieb category is a **k** linear pivotal category. It has one generating object X, generating morphisms cup cap and identity and relation

snakes (X is self-dual) + bubble (trace) = β.

$End(X^{\otimes n}) = \mathsf{TL}_n(\beta)$

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[1] *Linear algebraic groups*.

[2] D Ridout and Y Saint-Aubin. Standard Modules, Induction and the Temperley-Lieb Algebra. *Adv. Theor. Math. Phys.*, 18:957–1041, 2014.

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