Kleine Seminar: 13-10-2020

Getting back in categorical shape

Categories: $\mathcal{C}$ is a category if it has:

1: a class of objects $ob\,\mathcal{C}$

2: for $X, Y \in ob\,\mathcal{C}$ a set of morphisms $Hom(X,Y)$

3: $Id_X \in Hom(X,X)$ for every X, and a composition law $\circ : Hom(X,Y) \times Hom(Y,Z) \to Hom(X,Z)$ that makes $(Hom, \circ, id)$ a monoid.

Functors $F: \mathcal{C} \to \mathcal{D}$ sends object to object and morphism to morphism, reversing the direction of arrows if it is contravariant and keeping it if it is covariant.

A functor is full if $f \longmapsto F(f)$ is surjective, it is faithful if $f \longmapsto F(f)$ is injective, and fully faithful if it is both.

Abelian categories: A category $\mathcal{A}$ is abelian if

1) $\mathrm{Hom}(X,Y)$ is an abelian group and composition is bilinear
   Preadditive
2) It has direct sum and direct product, and a zero object
   additive
3) Every morphism has a kernel and a cokernel
   Preabelian
4) Every mono is a kernel and every epi, a cokernel.
   Abelian.

$\rightsquigarrow$ Strict + Weak $\Longleftrightarrow$ strict by McLane.

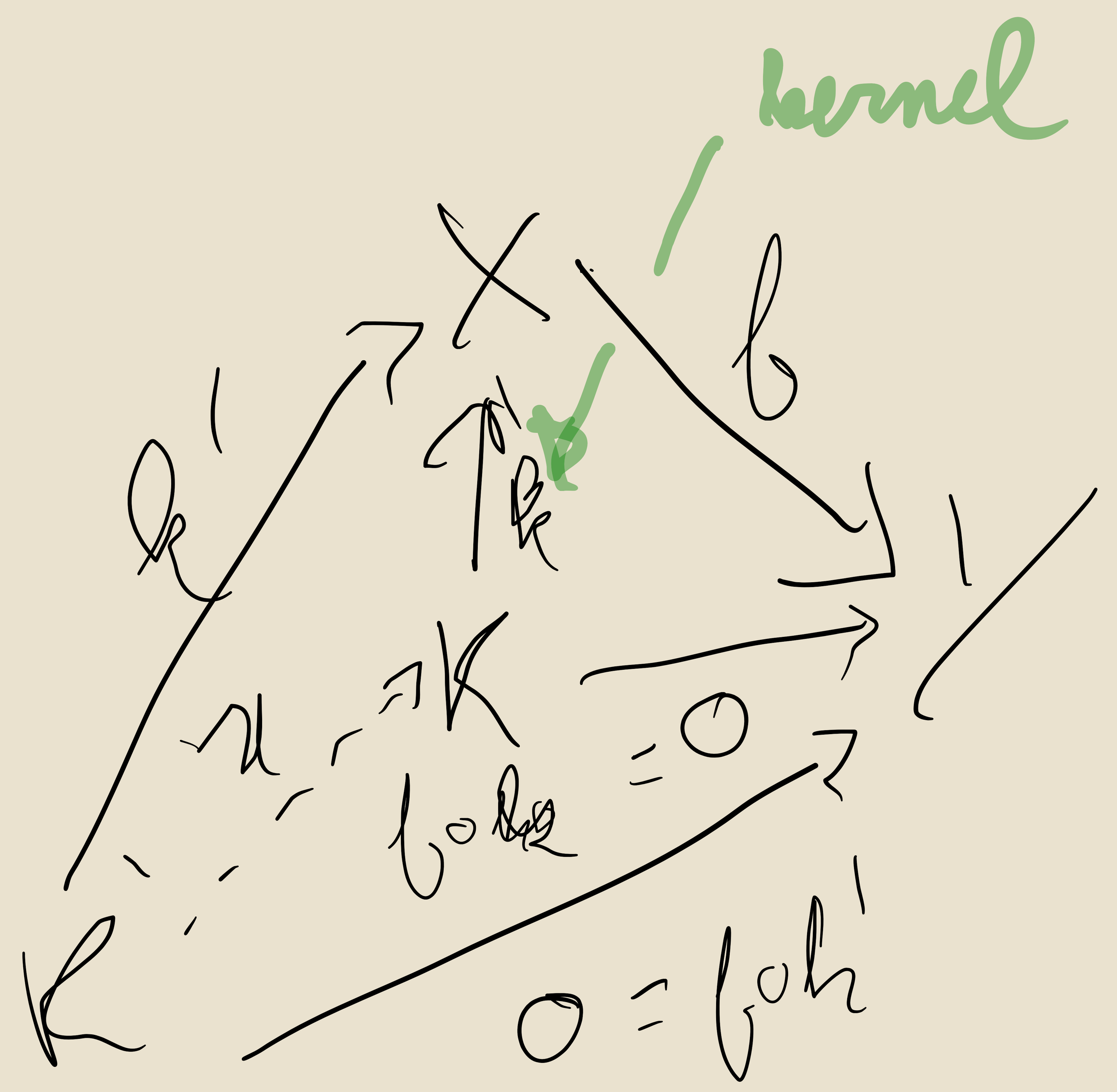
Monoidal categories: $\mathcal{M}$ is a monoidal category if it has

a) bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, the tensor product
b) a unit object $\mathbb{1}$

such that for every triple (X, Y, Z) the tensor product is associative. (weak $\Rightarrow$ up to iso)

More if needed.

Kernel and cokernel.



For a morphism $X \xrightarrow{b} Y$ a pair (K, k) a kernel if $b \circ k = 0$ and for every $k' : K' \to X$ such that $b \circ k' = 0$ the diagram commutes for a unique $u : K' \to K$.

More on categories

Cokernel: "reversing" the arrow

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "0"'] \arrow[ddr, bend right, "g"'] & Y \arrow[d, "\mathrm{coker}"] \arrow[dd, bend left] \\
& \mathrm{Cokernel} \arrow[d, dashed, "u"] \\
& C'
\end{tikzcd}
$$

2-categories

A 2-category $\mathcal{K}$ consists of:

1) objects $X, Y, Z \ldots$

2) For each pair of objects X, Y, a category $K(X,Y)$

2.1) The objects of $K(X,Y)$ are morphisms $X \xrightarrow{f} Y$ called 1-morphisms of $\mathcal{K}$. The morphisms of $K(X,Y)$ are $f \overset{\alpha}{\Longrightarrow} g$, called 2-morphisms

We denote them:

$$\begin{tikzcd} X \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] & Y \end{tikzcd}$$

2.2) A compatibility structure.

i) A composition for objects of $K(X,Y)$, so for 1-morph
$$X \xrightarrow{f} Y \xrightarrow{g} Z = X \xrightarrow{gf} Z.$$

ii) Two compositions for 2-morphisms

- Horizontal composition: $*$

$$\begin{tikzcd} X \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] & Y \arrow[r, bend left=50, "g'"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Uparrow\beta"] & Z \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left=50, "g'g"] \arrow[r, bend right=50, "f'f"'] \arrow[r, phantom, "\Uparrow\beta*\alpha"] & Z \end{tikzcd}$$

- Vertical composition: $\circ$

$$\begin{tikzcd}[row sep=large] X \arrow[r, bend left=70, "h"] \arrow[r, "g" description] \arrow[r, bend right=70, "f"'] \arrow[r, phantom, shift left=12, "\Uparrow\beta"] \arrow[r, phantom, shift right=12, "\Uparrow\alpha"] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left=50, "h"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Uparrow\beta\circ\alpha"] & Y \end{tikzcd}$$

2.3) The compositions are associative and have a unit $b \xRightarrow{1_b} b$; $X \xrightarrow{1_X} X$

→ other side

$$\begin{tikzcd} X \arrow[r, bend left=60, "b"] \arrow[r, "b" description] \arrow[r, bend right=60, "b"'] \arrow[r, phantom, shift left=3, "\Uparrow\alpha"] \arrow[r, phantom, shift right=3, "\Uparrow 1"] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left=50, "b"] \arrow[r, bend right=50, "b"'] \arrow[r, phantom, "\Uparrow\alpha"] & Y \end{tikzcd} = \dots$$

$$X \xrightarrow{1} X \xrightarrow{b} Y = X \longrightarrow Y = \dots$$

3) The interchange law $(\delta\cdot\gamma)*(\beta\alpha) = (\delta*\beta)\cdot(\gamma*\alpha)$

$$\begin{tikzcd} X \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Uparrow\beta"] \arrow[r, phantom, shift right=3, "\Uparrow\alpha"] & Y \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Uparrow\delta"] \arrow[r, phantom, shift right=3, "\Uparrow\gamma"] & Z \end{tikzcd}$$

"makes sense"

$$\begin{tikzcd} X \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Uparrow\beta\alpha"] & Y \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Uparrow\delta\gamma"] & Z \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Uparrow\delta*\beta"] \arrow[r, phantom, shift right=3, "\Uparrow\gamma*\alpha"] & Z \end{tikzcd}$$

Monoidal categories as 2-category

Let $\mathcal{M}$ be a (strict) monoidal category. Then we define $\Sigma(\mathcal{M})$, a 2-category version of $\mathcal{M}$ as follows:

$\Sigma(\mathcal{M})$	$\mathcal{M}$
Object(s)	$*$ an object (void)
1-morphisms	objects of $\mathcal{M}$
composition of 1-morphism	$\otimes$ of objects in $\mathcal{M}$.
2-morphisms	Morphisms of $\mathcal{M}$
horizontal composition	Tensor product of morphism
Vertical composition	composition of morphism
Interchange law	compatibility of $\otimes$ and composition of morphisms.

⚠ We defined strict 2-categories, a weak 2-cat. is when the interchange law is upto isomorphism. For each weak 2-category there is one equivalent strict 2-category.

R/Q Are all 2-categories with 1 object monoidal categories?

Conj/? Very likely

Yes

Categorification, Abelian side

Grothendieck group.

Let $\mathcal{C}$ be an abelian category. The Grothendieck group $K(\mathcal{C})$ of $\mathcal{C}$ is:

$$K(\mathcal{C}) = \langle X \in \mathrm{ob}\,\mathcal{C} \mid X = Y + Z \text{ if } 0 \to Y \to X \to Z \to 0 \text{ is exact} \rangle$$

Example. Let $\mathbb{C}$-Vec be the category of $\mathbb{C}$ vec space. Then $K(\mathcal{C}) \simeq \mathbb{Z}$.

$$\hookrightarrow 0 \to V_1 \to V_2 \to V_3 \to 0$$

$$V_2 \simeq V_1 \oplus V_3$$

if $\dim V = \dim W$, then $V \simeq W$.

Exact functor $F: \mathcal{C} \to \mathcal{D}$ is exact if it preserves exact sequences.

If $F: \mathcal{C} \to \mathcal{D}$ is exact then it induces a morphism of Grothendieck group

$$K(\mathcal{C}) \xrightarrow{[F]} K(\mathcal{D})$$

Weak categorification. We want to categorify a A-module.

Let A be a ring, $a = \{a_k\}_{k \in I}$ a basis of A such that

$$a_i a_j = \sum_{k \in I} c_{ij}^k a_k \quad , \quad c_{ij}^k \in \mathbb{N}_{\geq 0}$$

Let B be a A-module

[Taken from Khovanov, Mazorchuk, Stroppel, 2009 Intro to abelian categorification.]

Def A (weak) abelian categorification $\mathcal{C}(A, a, B)$ is an abelian category $\mathcal{C}$, an isomorphism $\varphi : K(\mathcal{C}) \to B$ and exact functors $F_i : \mathcal{C} \to \mathcal{C}$ such that

$$\begin{tikzcd} K(\mathcal{C}) \arrow[r, "{[F_i]}"] \arrow[d, "\varphi"'] & K(\mathcal{C}) \arrow[d, "\varphi"] \\ B \arrow[r, "a_i"] & B \end{tikzcd}$$

Commutes.

Furthermore $F_i F_j \cong \bigoplus_k F_k^{c_{ij}^k}$ taken from a.

Weyl algebra

$$A := \mathbb{Z}\langle x, \partial \rangle / (\partial x - x\partial - 1)$$

$\partial ::\ \{x^i \partial^j\}_{i,j \geq 0}$

$B =$ generated by $\left\{ \frac{x^n}{n!} \right\}_{n \geq 0}$, $B \subset \mathbb{Q}[x]$

Let R_n be the algebra generated by $Y_1, \dots, Y_{n-1}$ and relation

$$Y_i^2 = 0$$

$$Y_i Y_j = Y_j Y_i \qquad |i-j| > 1$$

$$Y_i Y_{i+1} Y_i = Y_{i+1} Y_i Y_{i+1}$$

The NilCoxeter algebra. For each R_n there is a unique simple module L_n.

Take $\mathcal{C} = \bigoplus_{n \geq 0} R_n\text{-Mod}$

the direct sum of categories of fin-dim modules of R_n

Then $K(R_n) \simeq \mathbb{Z}$

The R_n form a tower of algebras

$$R_1 \hookrightarrow R_2 \hookrightarrow \dots \hookrightarrow R_n \hookrightarrow R_{n+1} \hookrightarrow \dots$$

Take the restriction and induction morphism of algebra and lift them to functor

$$X_n : {}_{R_n}\mathrm{Mod} \longrightarrow {}_{R_{n+1}}\mathrm{Mod}$$

$$D_n : {}_{R_n}\mathrm{Mod} \longrightarrow {}_{R_{n-1}}\mathrm{Mod}$$

Then the X_n and the D_n will be what is necessary to categorify the Weyl algebra.

Take $\varphi : K(\mathcal{C}) \longrightarrow B$

$$L_n \longmapsto \frac{x^n}{n!}$$

L_n - the simple module of R_n

Basis elements $x^i \partial^j \longrightarrow X^i D^j$

And you have $DX \simeq XD \oplus \mathrm{id}$

Remark You can q-degrade them.