# COXETER GROUPS.

PLAN: 1) Coxeter Systems and Groups.
2) Reflection Representation.
3) Coxeter combinatorics.
4) Affine Reflection Groups.
5) Coxeter complex.

## 1) Coxeter Systems and Groups

Def A Coxeter System is pair $(W, S)$ s.t.

- $W$ a group
- $S \subset W$ a set of generators
- $\exists\, M : S \times S \longrightarrow \mathbb{N} \cup \{\infty\}$
  $(s,t) \longmapsto m_{st}$

  with $m_{ss} = 1$, $m_{st} > 1$ if $s \neq t$,
- $W = \langle s \in S \mid (st)^{m_{st}} = 1,\ \forall s,t \text{ with } m_{st} < \infty \rangle$

The grp $W$ is a Coxeter group, $|S|$ is the rank.
The Coxeter graph $\Gamma(W,S)$ is defined via:

- $S$ is the vertex set
- conect $s,t$ by a labelled edge if $m_{st} > 2$. $\square$

**E.g.:** $I_2(m)$ : $\overset{s}{\circ}\overset{m}{\text{———}}\overset{t}{\circ}$ $(m>2)$

**Claim**: $W(I_2(m)) \xleftarrow{\varphi} D_{2m} = \langle \sigma, \rho \mid \sigma^2 = \rho^m = 1, \sigma\rho\sigma = \rho^{-1} \rangle$

$$s \longleftarrow\!\shortmid\ \sigma$$
$$(st) \longleftarrow\!\shortmid\ \rho$$

is an isomorphism.

**Pf:** The inverse is $\psi(s) = \sigma$, $\psi(t) = \sigma\rho$.

$$\left.\begin{cases} \varphi(\sigma)^2 = s^2 = 1 = (st)^m = \varphi(\rho^m) \\ \varphi(\sigma)\varphi(\rho)\varphi(\sigma) = ts = \varphi(\rho)^{-1} \end{cases}\right\} \Rightarrow \varphi \text{ a hom.}$$

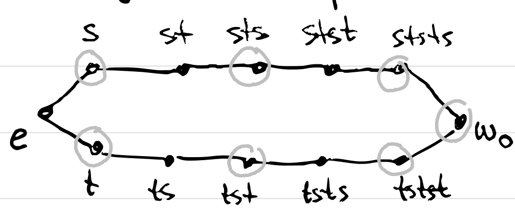
$$\left.\begin{cases} \psi(s)^2 = \sigma^2 = 1 = (\sigma\rho\sigma)\rho = \psi(t)^2 \\ (\psi(s)\psi(t))^m = \rho^m = 1 \end{cases}\right\} \Rightarrow \psi \text{ a hom.}$$

$\varphi\circ\psi$: $s \mapsto s$, $t \mapsto sst = t$ $\qquad$ $\psi\circ\varphi$: $\sigma \mapsto \sigma$, $\rho \mapsto \sigma\sigma\rho = \rho$ $\square$

**Rem.:** $\{\text{Cox. Systems}\} \xrightarrow{\pi} \{\text{Grps}\}$

$$(W, S) \longmapsto W$$

is not an injective map: consider $W(\underset{s}{\circ}\overset{6}{—}\underset{t}{\circ})$



Then, the maps below are each other inverses, and homs.

$$\varphi : W\big(\underset{\sigma}{\circ}\overset{3}{\text{—}}\underset{\tau}{\circ}\quad\underset{\rho}{\circ}\big) \longrightarrow W\big(\underset{s}{\circ}\overset{6}{\text{—}}\underset{t}{\circ}\big)$$

$$\sigma \longmapsto s$$

$$\tau \longmapsto (tst)$$

$$\rho \longmapsto w_0$$

$$\psi : W\big(\circ\overset{6}{\text{—}}\circ\big) \longrightarrow W\big(\circ\overset{3}{\text{—}}\circ\quad\circ\big)$$

$$s \longmapsto \sigma$$

$$t \longmapsto \sigma\tau\sigma\rho$$

$\square$

## 2) Reflection Representation

Let $(W, S)$ be given. Define

$$V = \mathbb{R}^{|S|} = \bigoplus_{s\in S} \mathbb{R}\alpha_s ,$$

$\{\alpha_s \,;\, s\in S\}$ a given basis of $V$.

Define $(\cdot,\cdot)$ on $V$ via

$$(*) \qquad (\alpha_s, \alpha_t) := -\cos\left(\frac{\pi}{m_{st}}\right)$$

If $s\in S$, let $\rho(s)v = v - 2(v,\alpha_s)\alpha_s$.

**Prop:** If $W$ finite, $\rho$ extends to a faithful irrep
$$\rho : W \longrightarrow O((\cdot,\cdot), V) \subseteq GL(V) \qquad \square$$

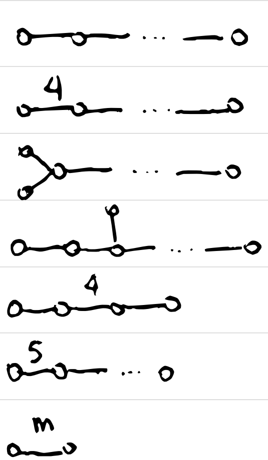
**Rem:** • $$(s(u), s(v)) = (u - 2(v,\alpha_s)\alpha_s,\ v - 2(\alpha_s, v)\alpha_s)$$
$$= (u,v) - 2\big((v,\alpha_s)(u,\alpha_s) + (u,\alpha_s)(v,\alpha_s)\big)$$
$$+ 4\,(u,\alpha_s)(v,\alpha_s)\underbrace{(\alpha_s,\alpha_s)}_{=1}$$
$$= (u,v)$$

• $\rho$ is called the reflection rep'n $\square$

**Thm:** Let $(W,S)$ be a C.S. Then

(1) $W$ finite $\Leftrightarrow$ $(\cdot,\cdot)$ (*) is pos. def.

(2) If $\Gamma =: \Gamma(W,S)$ connected, $W$ is finite iff $\Gamma$ in



C { $A_n$ $(n \geq 1)$; $B_n$ $(n \geq 2)$; $D_n$ $(n \geq 4)$; $E_n$ $(n = 6,7,8)$ }

$F_4$

$H_n$ $(n = 3,4)$

(C) $(m = 3,4,6)$ $I_m$ $(m \geq 3)$ $\square$

**Rmk:** $(W,S)$ Crystallogr. if $m_{st} \in \{2,3,4,6,\infty\}$, $s \neq t$. $\square$

## 3) Coxeter Combinatorics

Given $(W,S)$, an <u>expression</u> for $w \in W$ is

$(+)$ $\underline{w} = (s_1, \dots, s_k)$ $(s_j \in S)$

s.t. $w = \pi(\underline{w})$.

Def: If $\underline{w}$ as in $(+)$ $\Rightarrow$ $\ell(\underline{w}) = k$. Define

$$\ell(w) = \min\{\ell(\underline{w})\ ;\ \underline{w} \text{ an expression}\}$$

If $\ell(\underline{w}) = \ell(w)$ $\Rightarrow$ $\underline{w}$ is a <u>rex</u>. $\square$

Claim: The map $S \ni s \mapsto (-1)$ extends to a hom

$$\sigma : W \longrightarrow \{\pm 1\} \subseteq \mathbb{C}^\times$$

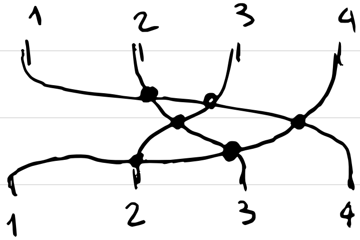
Pf: $\sigma(s)^2 = 1 = (\sigma(s)\sigma(t))^{m_{st}}$ $\square$

Cor: If $\underline{w}_1, \underline{w}_2$ are expressions for $w$

$$\Rightarrow \begin{cases} \ell(\underline{w}_1) \equiv \ell(\underline{w}_2) & (\text{mod } 2). \\ \ell(ws) \neq \ell(w) & , \forall w \in W, s \in S \end{cases}$$ $\square$

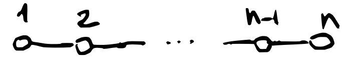
E.g.: In type A: 3 ways to represent $w \in W$.

1) $w = (4,3,2,1)$ $\quad w_i = w(i)$
2) $w = (14)(23)$

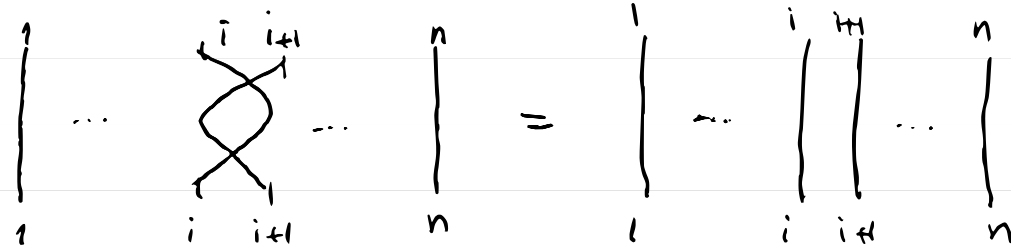
3) 

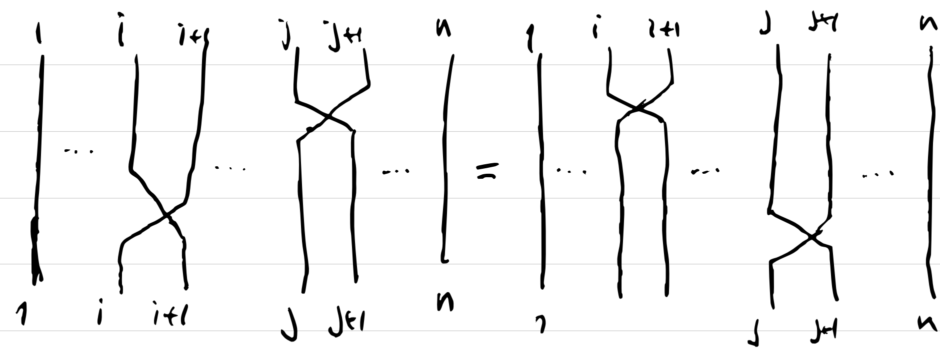

$\ell(w) = \#$ crossings.

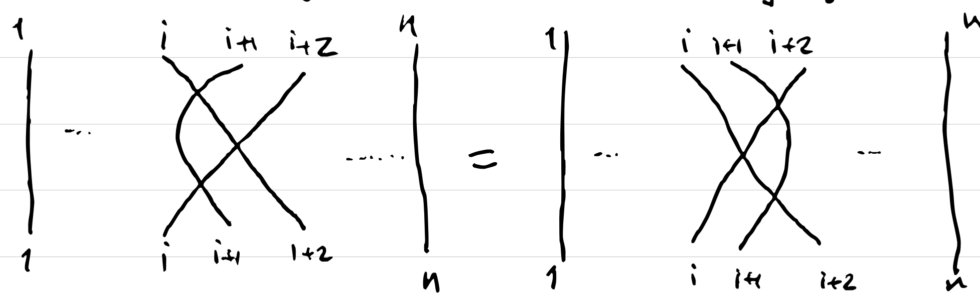
NB: $\underline{w} = (s_1, s_2, s_1, s_3, s_2, s_1)$ is a rex

In 3), the relations of 

are

$s_i^2 = 1$: 


$(s_i s_j)^2 = 1$, $|i-j|>1$: 


$(s_i s_j)^3 = 1$, $|i-j|=1$: 


□

**Prop:** (Properties of $\ell(w)$):

1) $\ell(w) = 1 \iff w \in S$
2) $\ell(w) = \ell(w^{-1})$
3) $\ell(w) - \ell(w') \leq \ell(ww') \leq \ell(w) + \ell(w')$
4) $\ell(ws) \in \{\ell(w) \pm 1\}$
5) $\underline{w} = (s_1, \dots, s_k)$ rex and $\ell(ws) \leq \ell(w)$ (EC)

$\Rightarrow \exists i$ s.t. $wt = s_1 \dots \hat{s}_i \dots s_k$

6) $\underline{w} = (s_1, \dots, s_k)$ and $\ell(w) < \ell(\underline{w}) = k$, (DC)

$\Rightarrow \exists i < j$ s.t. $w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$ $\square$

**Rem:**
- from $\ell(ww') \leq \ell(w) + \ell(w')$ and 2), get
$\ell(w) = \ell(ww'(w')^{-1}) \leq \ell(ww') + \ell(w')$.
- 6) is a consequena of 5)
- 6) <u>characterizes</u> a Cox. System:
if $(G, S)$ a group generated by involutions $S$
$\Rightarrow$ DC $\Rightarrow (G, S)$ a CS. $\square$

**Def:** (Bruhat Order)

- $x \to y$ if $\ell(x) < \ell(y)$ & $y = xt$, $\exists t \in T = \bigcup_w wSw^{-1}$
- <u>Bruhat order</u> is the trans closure of $\to$.
- <u>Bruhat Graph</u>: vertex $W$, edges according to $\to$. $\square$

E.g.: In $I_2(3)$:

$$\begin{tikzcd}
 & s \arrow[r] \arrow[dr] & st \arrow[dr] & \\
e \arrow[ur] \arrow[dr] & & & w_0 \\
 & t \arrow[r] \arrow[ur] & ts \arrow[ur] &
\end{tikzcd}$$

$\square$

## 4) Affine reflection groups

Let $(V, B)$ be Euclidean space.

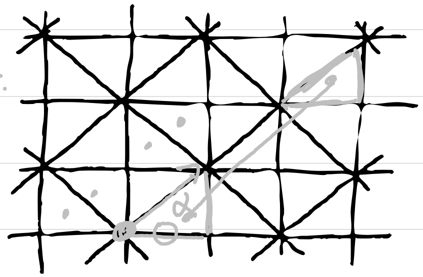
Recall:

- $s \in O(V,B)$ reflection:
$$\begin{cases} rk(1-s) = 1 \\ H_s = \{ v \mid s(v) = v \} = \alpha_s^{\perp}, \ \exists \alpha_s \in V \end{cases}$$
- $T_v : V \longrightarrow V : u \longmapsto u + v$ is a translation
- $A \in Aff(V) \iff A(v) = f(v) + u = T_u \circ f(v)$, $\exists f \in GL(V)$

Def.: • $\sigma$ is an <u>aff. reflection</u> if $\sigma = T_v \circ s \circ T_{-v}$ with $s$ a reflection, $v \in V$

• $W \subseteq Aff(V)$ is an <u>affine reflection group</u> if
  a) $W$ gen. by aff. refl.;
  b) $W$ is <u>proper</u>: $\forall K, L$ cpct, $|K \cap wL| < \infty$, $\forall w$ $\square$

Rmk b) $\Rightarrow$ discrete orbits.

Eg.: 

$V = \mathbb{R}^2$

$W = \langle \text{refl. over mirrors} \rangle$

$\square$

Given $H = H_{(\alpha,n)} = \{ v \mid (\alpha, v) = n \}$,

$s_H = T_{n\alpha} \circ s_\alpha \circ T_{-n\alpha}$,

$v \longmapsto n\alpha + s_\alpha(v - n\alpha) = s_\alpha(v) + 2n\alpha$

Let $\Phi = \{ H ; \exists s \in W \text{ with } s = s_H \}$

(b) $\Rightarrow$ $\Phi$ locally finite

$\Rightarrow$ $V \setminus \bigcup_{H \in \Phi} H$ is open.

Let $\mathcal{A} := \pi_0 (V \setminus \bigcup_{\Phi} H)$

$\bar{\mathcal{A}} := \{ \bar{A} ; A \in \mathcal{A} \}$

$A \in \mathcal{A}$ is called an alcove

Choose $\Delta \in \mathcal{A}$,

$\Phi_\Delta = \{ H \in \Phi ; | H \cap \bar{\Delta} | > 1 \}$

$S = \{ s_H ; H \in \Phi_\Delta \}$

$W \supseteq W_S = \langle S \rangle$

Thm:

1) $W_S$ acts trans. on $\mathcal{A}$
2) $W = W_S$
3) $H_{(\alpha,k)}, H_{(\beta,e)} \in \Phi_\Delta \Rightarrow \measuredangle(\alpha,\beta) = \pi/m, \exists m \in \mathbb{N}$
4) $(W,S)$ is a Coxeter system $\square$

Rem. • $\ell(w) = \#\{H \mid H$ separates $\Delta$ and $w(\Delta)$
• $\Gamma(W,S)$ connected $\Rightarrow \Gamma$ is the extension of a finite $(W_0,S)$, for a unique node. $\square$

5) Coxeter Complex.

Given $(W,S)$, with $n = |S| < \infty$

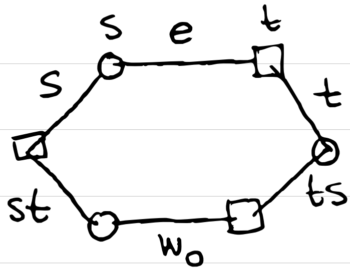
let • $\Delta \simeq \Delta^{n-1}$, the std simplex
• label its $n$ facets with $s \in S$

Def (Coxeter Cplx)

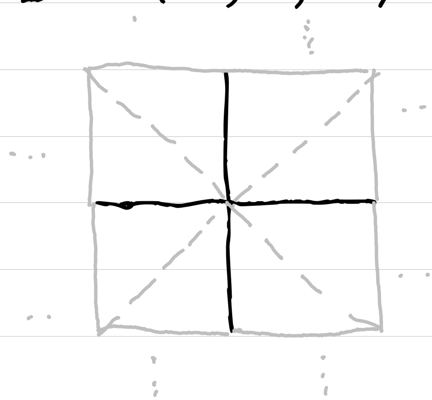
• $$\Delta(W,S) = \coprod_{w\in W} \Delta_w \Big/ \sim$$

$\sim$: we glue the points of $\Delta_w, \Delta_{ws}$ along the $s$-label $\square$

E.g. • $I_2(3)$:




• $\tilde{B}_2 = \langle s, t, u \rangle$:



$\square$