A_r : $V = \mathbb{R}^{r+1}$, $\Lambda = \Lambda_{sc}$

Simple roots:

$$\alpha_i = e_i - e_{i+1}, \quad i = 1, \dots, r$$

Fundamental weights:

$$\varpi_i = e_1 + \dots + e_i, \quad i = 1, \dots, r$$

Standard crystal $\mathbb{B} = B_{\varpi_1}$:

~~$wt(\boxed{1}) = e_1 = \varpi_1$~~

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \dots \xrightarrow{r} \boxed{r+1}$$

$$wt(\boxed{1}) = e_1 = \varpi_1$$

$$wt(\boxed{i}) = e_i$$

Alphabet:

$$\mathcal{A}_{A_r} = \{1 < 2 < \dots < r+1\}$$

~~$ch(u_\lambda) =$~~

~~Question~~

~~$B_\lambda \subset B_{\varpi_1}^{\otimes(\lambda_1-\lambda_2)} \otimes B_{\varpi_2}^{\otimes(\lambda_2-\lambda_3)} \otimes \dots \otimes B_{\varpi_{r-1}}^{\otimes(\lambda_{r-1}-\lambda_r)} \otimes B_{\varpi_r}^{\otimes \lambda_r} \subset \mathbb{B}^{\otimes \lambda}$~~

Fundamental crystals B_{ϖ_k}, $k = 1, \dots, r$:

For $k = 1, \dots, r$

$B_{\varpi_k} \subset \mathbb{B}^{\otimes k}$: The crystal generated by the highest weight element

$$\boxed{k} \otimes \dots \otimes \boxed{1} \in \mathbb{B}^{\otimes k}$$

Crystals of tableaux:

Let λ be a dominant weight, then $\lambda = \sum_{i=1}^{r} \lambda_i e_i$, $\lambda_1 \geq \dots \geq \lambda_r \geq 0$.

We want ~~a~~ a crystal B_λ generated from a h.w. element u_λ of weight $wt(u_\lambda) = \lambda$.

~~Let~~ Set of tableaux of type A_r:

Tab_λ = { Tableaux of shape λ, in alphabet $\mathcal{A}_{A_r}$ s.t. rows weakly increasing and columns strictly increasing }

= { Semistandard Young tableaux in alphabet $\mathcal{A}_{A_r}$ of shape λ }

Crystal structure on Tab_λ:

Rows: ~~If~~ $1 \leq j_1 \leq \dots \leq j_k \leq r$, then

$$\underbrace{RR}_{=CR}\left(\boxed{j_1}\boxed{j_2}\boxed{\dots}\boxed{j_k}\right) := \boxed{j_1} \otimes \dots \otimes \boxed{j_k} \in \mathbb{B}^{\otimes k}$$

Columns: If $1 \leq j_1 < \dots < j_k \leq r$, then

$$CR/RR\begin{pmatrix} \boxed{j_1} \\ \vdots \\ \boxed{j_k} \end{pmatrix} := \boxed{j_k} \otimes \dots \otimes \boxed{j_1} \in \mathbb{B}^{\otimes k}$$

Row reading and column reading are equal on tableaux of hook shape

Let $T \in Tab_\lambda$ with rows $R_1, \dots, R_s$ and columns $C_1, \dots, C_t$, then

Row reading: $RR(T) \in \mathbb{B}^{\otimes |\lambda|}$, $RR(T) = RR(R_s) \otimes \dots \otimes RR(R_1) \in \mathbb{B}^{\otimes |\lambda|}$

Column reading: $CR(T) \in \mathbb{B}^{\otimes |\lambda|}$, $CR(T) = CR(C_1) \otimes \dots \otimes CR(C_t) \in \mathbb{B}^{|\lambda|}$

$(Tab_\lambda, RR) \simeq (Tab_\lambda, CR)$ as crystals

$B_\lambda \simeq Tab_\lambda \subset \mathbb{B}^{\otimes |\lambda|}$: Tab_λ with either RR or CR is a connected subcrystal of $\mathbb{B}^{\otimes |\lambda|}$ generated by the h.w. element $RR(u_\lambda)$ or $CR(u_\lambda)$ where u_λ is the Yamanouchi tableau of shape λ:

The tableaux with only i's in the i'th row.

$\lambda = (5, 3, 1)$: $u_\lambda =$

1	1	1	1	1
2	2	2		
3				

C_r : $V = \mathbb{R}^r$, $\Lambda = \Lambda_{sc}$

Simple roots:

$\alpha_i = e_i - e_{i+1}$, $i = 1, \dots, r-1$

$\alpha_r = 2e_r$

Fundamental weights:

$\overline{\omega}_i = e_1 + \cdots + e_i$, $i = 1, \dots, r$

Standard crystal $\mathbb{B} = B_{\overline{\omega}_1}$:

$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{\bar{r}} \xrightarrow{r-1} \cdots \xrightarrow{1} \boxed{\bar{1}}$

$wt(\boxed{1}) = e_1 = \overline{\omega}_1$

$$\left.\begin{array}{l} wt(\boxed{i}) = e_i , \\ wt(\boxed{\bar{i}}) = -e_i \end{array}\right\} i = 1, \dots, r$$

~~Goal:~~

~~• Construct B_λ inside a tensor product of fundamental crystals~~

~~$B_\lambda \subset B_{\overline{\omega}_1}^{\otimes(\lambda_1-\lambda_2)} \otimes \cdots \otimes B_{\overline{\omega}_{r-1}}^{\otimes(\lambda_{r-1}-\lambda_r)} \otimes B_{\overline{\omega}_r}^{\otimes \lambda_r}$~~

Alphabet:

$\mathcal{A}_{C_r} = \{1 < 2 < \cdots < r < \bar{r} < \cdots < \bar{2} < \bar{1}\}$

Fundamental crystals $B_{\overline{\omega}_k}$, $k = 1, \dots, r$:

$B_{\overline{\omega}_k} \subset \mathbb{B}^{\otimes k}$: The crystal generated from the highest weight element
$k = 1, \dots, r$

$$\boxed{k} \otimes \cdots \otimes \boxed{1} \in \mathbb{B}^{\otimes k}$$

~~Crystals of tableaux:~~

~~Let λ be a dominant weight, that is, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$~~

~~Set of tableaux of type C_r~~

~~Alphabet: $\mathcal{A}_{C_r} = \{1 < 2 < \cdots < r < \bar{r} < \cdots < \bar{1}\}$~~

Crystals of columns:

$\mathcal{C}_k = \Big\{$ Columns of height k ~~that are~~ in the alphabet $\mathcal{A}_{C_r}$, that are
1) Strictly increasing from top to bottom
2) If both letters i and $\bar{i}$ appear in the column, and i is in the a'th box from the top and $\bar{i}$ is in the b'th box from the bottom, then $a + b \leq i$ $\Big\}$

With column reading $\begin{array}{|c|}\hline i_1 \\ \vdots \\ i_k \\ \hline\end{array} \mapsto \boxed{i_k} \otimes \cdots \otimes \boxed{i_1}$, $\mathcal{C}_k$ is a subcrystal of $\mathbb{B}^{\otimes k}$ and $\mathcal{C}_k \simeq B_{\overline{\omega}_k}$ $\forall k \in \{1, \dots, r\}$.

Crystals of tableaux: Type C_r:

Let λ be a dominant weight, that is $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$.

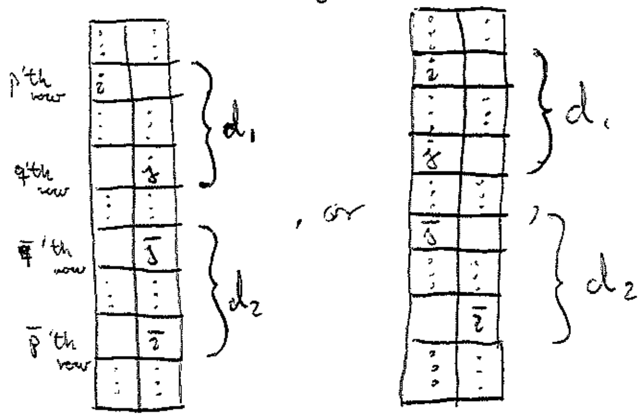
Set of tableaux of type C_r:

$Tab_\lambda = \{$ Tableaux T of shape λ in the alphabet $\mathcal{A}_{C_r}$ such that each col

C1) Each column of height k in T is in $\mathcal{C}_k$

C2) Each row is weakly increasing

C3) If T has two adjacent columns of either form,



$d_1 = |p-q|$

$d_2 = |\bar{p}-\bar{q}|$

Where $i \leq j$ and $d_1, d_2 \geq 0$ are vertical distances, then we must have $d_1 + d_2 < j - i$

$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline \bar{3} & \bar{2} \\ \hline \end{array}$ is not in $Tab_{(2,2,2)}$ since if $i=2, j=3$, then $d_1 = 1, d_2 = 0$ and $d_1 + d_2 = 1 \not< j - i = 1$

$\begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline 3 & \\ \hline \bar{3} & \\ \hline \end{array}$ is in $Tab_{(2,1,1)}$ $\longrightarrow$ Make crystal

$Tab_\lambda \subset \mathbb{B}^{\otimes|\lambda|}$ is a crystal.

$$Tab_\lambda \hookrightarrow \mathcal{C}_{k_1} \otimes \cdots \otimes \mathcal{C}_{k_\ell} \hookrightarrow \mathbb{B}^{\otimes k_1 + \cdots + k_\ell} = \mathbb{B}^{\otimes|\lambda|}$$

$$T \longmapsto C_1 \otimes \cdots \otimes C_\ell$$

~~where $k_i = \lambda_i - \lambda_{i+1}$ is the~~

where k_i is the height of the i'th column and

C_i is the i'th column

$T = C_1 \cdots C_\ell$ by concatenation of columns

Thm:

$$Tab_\lambda \simeq B_\lambda$$

Notes:

$$Tab_\lambda \hookrightarrow C_{k_1} \otimes \cdots \otimes C_{k_\ell} = C_r^{\otimes \lambda_r} \otimes C_{r-1}^{\otimes(\lambda_{r-1}-\lambda_r)} \otimes \cdots \otimes C_1^{\otimes(\lambda_1-\lambda_2)}$$

IS $\qquad$ IS

$$B_\lambda \qquad B_{\varpi_r}^{\otimes \lambda_r} \otimes B_{\varpi_{r-1}}^{\otimes(\lambda_{r-1}-\lambda_r)} \otimes \cdots \otimes B_{\varpi_1}^{\otimes(\lambda_1-\lambda_2)}$$

So Tab_λ is isomorphic to the sub crystal in generated by the highest weight element.

B_r : $V = \mathbb{R}^r$, $\Lambda = \Lambda_{sc}$

Simple roots:

$$\alpha_i = e_i - e_{i+1}, \quad i = 1, \ldots, r-1$$
$$\alpha_r = e_r$$

Fundamental weights:

$$\varpi_i = e_1 + \cdots + e_i, \quad i = 1, \ldots, r-1$$
$$\varpi_r = \tfrac{1}{2}(e_1 + \cdots + e_r)$$

Standard crystal $\mathbb{B} = B_{\varpi_1}$:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{0} \xrightarrow{r} \boxed{\bar{r}} \xrightarrow{r-1} \cdots \xrightarrow{1} \boxed{\bar{1}}$$

$$\mathrm{wt}(\boxed{1}) = e_1 = \varpi_1$$

$$\cancel{\mathrm{wt}(\boxed{\bar{1}}) = e}$$

$$\left.\begin{array}{l} \mathrm{wt}(\boxed{i}) = e_i \\ \mathrm{wt}(\boxed{0}) = 0 \\ \mathrm{wt}(\boxed{\bar{i}}) = -e_i \end{array}\right\} \quad i = 1, \ldots, r$$

Alphabet:

$$\mathcal{A}_{B_r} = \{1 < 2 < \cdots < r < 0 < \bar{r} < \cdots < \bar{2} < \bar{1}\}$$

Fundamental crystals B_{ϖ_k}, $k = 1, \ldots, r$:

For $k = 1, \ldots, r-1$:

$B_{\varpi_k} \subset \mathbb{B}^{\otimes k}$: The crystal generated from the highest weight element
$$\boxed{k} \otimes \cdots \otimes \boxed{1} \in \mathbb{B}^{\otimes k}$$

For $k = r$:

$B_{2\varpi_r} \subset \mathbb{B}^{\otimes r}$; The crystal generated from the h.w. element
$$\boxed{r} \otimes \cdots \otimes \boxed{1} \in \mathbb{B}^{\otimes r}$$

B_{ϖ_r} cannot be found in $\mathbb{B}^{\otimes k}$ for any k!

Instead (1) B_{ϖ_r} can be realized as the virtual crystal V inside the crystal $B_{\varpi_r} \otimes B_{\varpi_{r+1}}$ of type D_{r+1} generated by $u_{\varpi_r} \otimes u_{\varpi_{r+1}}$ and the crystal operators $f_i = \hat{f}_i^2$, $i = 1, \ldots, r-1$, $f_r = \hat{f}_r \hat{f}_{r+1}$.

(2) B_{ϖ_r} is isomorphic to the "minuscule" crystal M_{ϖ_r} of type B_r.

Ex. $B_{\varpi_3} \cong M_{\varpi_3}$ of type B_3:

$$\begin{tikzcd}[column sep=small, row sep=small]
& & & \boxed{+--} \arrow[dr, "1"] & & & \\
\boxed{+++} \arrow[r, "3"] & \boxed{++-} \arrow[r, "2"] & \boxed{+-+} \arrow[ur, "3"] \arrow[dr, "1"'] & & \boxed{-+-} \arrow[r, "2"] & \boxed{--+} \arrow[r, "3"] & \boxed{---} \\
& & & \boxed{-++} \arrow[ur, "3"'] & & &
\end{tikzcd}$$

~~Crystals of tableaux:~~

~~Let λ be a dominant weight, that is,~~

Crystals of columns:

$\mathcal{C}_k$ = Columns of height k ~~with~~ in the alphabet $\mathcal{A}_{B_r}$, that are

1) Strictly increasing from top to bottom, ~~with~~ except
   (a) the letter 0 can be repeated
2) If both j and $\bar{j}$ appear in the column, and j is in the a'th box from the top and $\bar{j}$ is in the b'th column from the bottom, then $a + b \leq j$

Thm:
With column reading $\begin{array}{|c|} \hline j_1 \\ \hline \vdots \\ \hline j_k \\ \hline \end{array} \mapsto \boxed{j_k} \otimes \cdots \otimes \boxed{j_1}$, $\mathcal{C}_k$ is a subcrystal of $\mathbb{B}^{\otimes k}$ and $\mathcal{C}_k \cong B_{\varpi_k}$ for $k = 1, \ldots, r-1$ and $\mathcal{C}_r \cong B_{2\varpi_r}$

Crystals of tableaux of type B_r:

Let λ be a ~~crystal of~~ dominant weight, that is $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$ and ~~either $\lambda_i \in \mathbb{Z}$ or $\lambda_i \in \mathbb{Z}+\frac{1}{2}$~~

(1) $\lambda_i \in \mathbb{Z}$, $i=1,\dots,r$ $\rightsquigarrow$ represented by a Young diagram ~~shape~~

(2) $\lambda_i \in \mathbb{Z}+\frac{1}{2}$, $i=1,\dots,r$ $\rightsquigarrow$ — // — with ~~added~~ a column of height r and width $\frac{1}{2}$ added to the front of the diagram.

Spin crystal:

$$\mathcal{C}_r^{spin} = \left\{ \begin{array}{l} \text{Columns of height } r \text{ and width } \tfrac{1}{2} \text{ filled increasingly with} \\ \text{letters } 1,\dots,r \ \text{~~each appearing exactly~~} \\ \text{each appearing exactly once either barred or} \\ \text{unbarred} \end{array} \right\}$$

$$\mathcal{C}_r^{spin} \xrightarrow{\sim} \mathcal{M}_{\bar{\omega}_r} \simeq B_{\bar{\omega}_r}$$

$$\boxed{\begin{array}{c} j_1 \\ \vdots \\ j_r \end{array}} \longmapsto \boxed{\epsilon_1 \cdots \epsilon_r}, \quad \epsilon_k = \begin{cases} + , & \text{if } k = j_s \text{ for some } s \\ - , & \text{if } \bar{k} = j_s \text{ for some } s \end{cases}$$

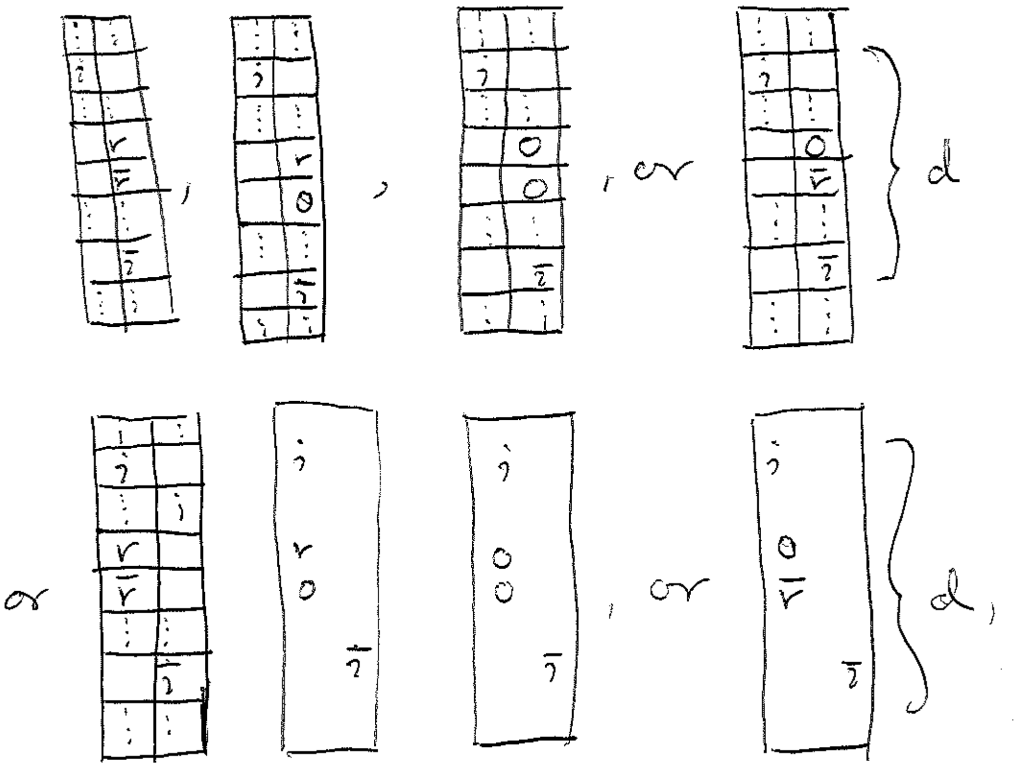
Tab_λ = Tableaux T of shape λ in alphabet A_{B_r} s.t.

B1) Each column of width 1 and heigth k is in $\mathcal{C}_k$ and each column of width $\frac{1}{2}$ and heigth r is in $\mathcal{C}_r^{spin} \simeq \mathcal{M}_{\bar{\omega}_r}$

B2) Each row in T is weakly increasing, but 0 cannot be repeated

B3) Condition C3 holds for $1 \leq i \leq j < r$.

B4) If T has two adjacent columns of the form



where $i \leq r$ and $d > 0$ is the vertical distance between i and $\bar{i}$, then we must have $d-1 < r-i$.

Thm: $\mathrm{Tab}_\lambda \simeq B_\lambda$

Notes:

Integer weights:
$$\mathrm{Tab}_\lambda \hookrightarrow \mathcal{C}_r^{\otimes \lambda_r} \otimes \mathcal{C}_{r-1}^{\otimes(\lambda_{r-1}-\lambda_r)} \otimes \cdots \mathcal{C}_1^{\otimes(\lambda_1-\lambda_2)} \simeq B_{2\bar{\omega}_r}^{\otimes \lambda_r} \otimes B_{\bar{\omega}_{r-1}}^{\otimes \lambda_{r-1}-\lambda_r} \otimes \cdots \otimes B_{\bar{\omega}_1}^{\otimes \lambda_1-\lambda_2}$$

Half integer weights:
$$\mathrm{Tab}_\lambda \hookrightarrow \mathcal{C}_r^{spin} \otimes \mathcal{C}_r^{\otimes \lambda_r-\frac{1}{2}} \otimes \mathcal{C}_{r-1}^{\otimes(\lambda_{r-1}-\lambda_r)} \otimes \cdots \otimes \mathcal{C}_1^{\otimes(\lambda_1-\lambda_2)} \simeq B_{\bar{\omega}_r} \otimes B_{2\bar{\omega}_r}^{\otimes \lambda_r} \otimes B_{\bar{\omega}_{r-1}}^{\otimes \lambda_{r-1}-\lambda_r} \otimes \cdots \otimes B_{\bar{\omega}_1}^{\otimes \lambda_1-\lambda_2}$$

So Tab_λ is isomorphic to the subcrystal of or generated by the highest weight element.

D_r: $V=\mathbb{R}^r$, $\Lambda=\Lambda_{sc}$:

Simple roots:

$$\alpha_i = e_i - e_{i+1}, \quad i=1,\dots,r-1$$
$$\alpha_r = e_{r-1}+e_r$$

Fundamental weights:

$$\varpi_i = e_1+\dots+e_i, \quad i=1,\dots,r-2$$
$$\varpi_{r-1} = \tfrac{1}{2}(e_1+\dots+e_{r-1}-e_r)$$
$$\varpi_r = \tfrac{1}{2}(e_1+\dots+e_{r-1}+e_r)$$

Standard crystal $\mathbb{B}=B_{\varpi_1}$:

$$\begin{tikzcd}
 & & & \boxed{r} \arrow[dr, "r"] & & & \\
\boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "r-2"] \; \boxed{r-1} \arrow[ur, "r-1"] \arrow[dr, "r"'] & & \boxed{\overline{r-1}} \arrow[r, "r-2"] & \cdots \arrow[r, "1"] & \boxed{\bar{1}} \\
 & & & \boxed{\bar{r}} \arrow[ur, "r-1"'] & & &
\end{tikzcd}$$

$$wt(\boxed{1}) = e_1 = \varpi_1$$
$$\left.\begin{aligned} wt(\boxed{i}) &= e_i \\ wt(\boxed{\bar{i}}) &= -e_i \end{aligned}\right\} \quad i=1,\dots,r$$

Alphabet:

$$\mathcal{A}_{D_r} = \{1<2<\dots<r, \bar{r}<\overline{r-1}<\dots<\bar{2}<\bar{1}\}$$

↳ Here only r and $\bar{r}$ are incomparable.

Fundamental crystals B_{ϖ_k}, $k=1,\dots,r$:

For $k=1,\dots,r-2$:

$B_{\varpi_k} \subset \mathbb{B}^{\otimes k}$: The crystal generated from the highest weight element
$$\boxed{k}\otimes\dots\otimes\boxed{1} \in \mathbb{B}^{\otimes k}$$

For $k=r-1$:

$B_{2\varpi_{r-1}} \subset \mathbb{B}^{\otimes r}$: The crystal generated from the h.w. element
$$\boxed{\bar{r}}\otimes\boxed{r-1}\otimes\dots\boxed{1} \in \mathbb{B}^{\otimes r}$$

For $k=r$:

$B_{2\varpi_r} \subset \mathbb{B}^{\otimes r}$: The crystal generated from the h.w. element
$$\boxed{r}\otimes\dots\otimes\boxed{1} \in \mathbb{B}^{\otimes r}$$

~~To get B~~

$B_{\varpi_{r-1}}$ and B_{ϖ_r} cannot be found in $\mathbb{B}^{\otimes k}$ for any k!

Instead they can be realized as "minuscule" crystals $\underset{B_{\varpi_{r-1}}}{\underset{\cong}{M_{\varpi_{r-1}}}}$ and $\underset{B_{\varpi_r}}{\underset{\cong}{M_{\varpi_r}}}$.

~~Ex~~ – Both $B_{\varpi_{r-1}}$ and B_{ϖ_r} are Stembridge, because a miniscule crystal for a simply-laced root system is Stembridge.

Ex: $r=4$

$$B_{\varpi_3} \cong M_{\varpi_3}: \begin{tikzcd}
 & & & \boxed{+--+} \arrow[dr, "1"] & & & \\
\boxed{++++} \arrow[r, "4"] & \boxed{++--} \arrow[r, "2"] & \boxed{+-+-} \arrow[ur, "3"] \arrow[dr, "1"'] & & \boxed{-+-+} \arrow[r, "2"] & \boxed{--++} \arrow[r, "4"] & \boxed{--++} \\
 & & & \boxed{-++-} \arrow[ur, "3"'] & & &
\end{tikzcd}$$

$$B_{\varpi_4} \cong M_{\varpi_4}: \begin{tikzcd}
 & & & \boxed{+---} \arrow[dr, "1"] & & & \\
\boxed{+++-} \arrow[r, "3"] & \boxed{++-+} \arrow[r, "2"] & \boxed{+-++} \arrow[ur, "4"] \arrow[dr, "1"'] & & \boxed{-+--} \arrow[r, "2"] & \boxed{--+-} \arrow[r, "3"] & \boxed{---+} \\
 & & & \boxed{-+++} \arrow[ur, "4"'] & & &
\end{tikzcd}$$

Crystals of ~~tableaux~~ columns of type D_r:

$\mathcal{C}_k = \Big\{$ Columns of height k in the alphabet $\mathcal{A}_{D_r}$, that are

1) Strictly increasing from top to bottom, except
   b) the letters r and $\bar{r}$ in type D_r can alternate
2) If both j and $\bar{j}$ appear in the column, and j is in the a'th box from the top and $\bar{j}$ is in the b'th column from the bottom, then $a+b \le j$. $\Big\}$

$\mathcal{C}_r^+ = \Big\{$ Columns in $\mathcal{C}_r$ such that if r (or $\bar{r}$) appear in the column in the j'th column from ~~ab~~ above, then $r-j$ is even (or odd) $\Big\}$

$\mathcal{C}_r^- = \Big\{$ Columns in $\mathcal{C}_r$ such that if r (or $\bar{r}$) appear in the column in the j'th column from above, then $r-j$ is odd (or even) $\Big\}$

Ex. For type D_4, the element

$$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \bar{4} \\ \hline 4 \\ \hline \end{array} \in \mathcal{C}_4^+, \in \mathcal{C}_1^+, \notin \mathcal{C}_3^+ \quad \text{, whereas} \quad \begin{array}{|c|}\hline 1 \\ \hline \bar{4} \\ \hline 4 \\ \hline \bar{2} \\ \hline \end{array} \in \mathcal{C}_4^-$$

Thm:

With column reading $\begin{array}{|c|}\hline j_1 \\ \hline \vdots \\ \hline j_k \\ \hline \end{array} \longmapsto \boxed{j_k} \otimes \cdots \otimes \boxed{j_1}$,

$\mathcal{C}_k$, for $k=1,\dots,r$, $\mathcal{C}_r^+$ and $\mathcal{C}_r^-$ are connected crystals in $B^{\otimes k}$ and $B^{\otimes r}$.

In addition,

$$\mathcal{C}_k \simeq B_{\varpi_k},\ k=1,\dots,r-2,\ \mathcal{C}_{r-1} \simeq B_{\varpi_{r-1}+\varpi_r},\ \mathcal{C}_r^+ \simeq B_{2\varpi_{r-1}},\ \mathcal{C}_r^- \simeq B_{2\varpi_r}$$

Crystals of tableaux of type D_r:

Let λ be a dominant weight, that is,

$\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_{r-1} \ge |\lambda_r|$, and

(1) $\lambda \in \mathbb{Z}^r$

(2) $\lambda \in (\mathbb{Z}^* + \frac{1}{2})^r$

↳ λ can be viewed as a partition and with ~~possibly~~ a column of width $\frac{1}{2}$ and height r if $\lambda \in (\mathbb{Z}+\frac{1}{2})^r$

In particular if $\lambda = a_1\varpi_1 + \dots + a_r\varpi_r$, then λ has

width 1 $\Big\{$

↳ a_i columns of height i, for $i=1,\dots,r-2$

↳ $\bar{a}_{r-1} := \min\{a_{r-1}, a_r\}$ columns of height $r-1$

↳ $\bar{a}_r := \lfloor \frac{1}{2}(\max(a_{r-1},a_r) - \underbrace{\min(a_{r-1},a_r)}_{=\bar{a}_{r-1}}) \rfloor$ columns of height r

↳ 1 column of heigth r and width $\frac{1}{2}$, if $\max(a_{r-1},a_r) - \min(a_{r-1},a_r)$ is odd $\Big\{$ (1) zero such columns (2) 1 such column

Spin crystals:

$$\mathcal{C}_r^{\text{spin}-} \simeq B_{\varpi_{r-1}} \simeq \mathcal{M}_{\varpi_{r-1}}, \quad \mathcal{C}_r^{\text{spin}+} \simeq B_{\varpi_r} \simeq \mathcal{M}_{\varpi_r}$$

$\mathcal{C}_r^{\text{spin}\pm} = \Big\{$ Columns of height r and width $\frac{1}{2}$ filled with ~~the~~ each ~~letters~~ letter $1, \dots, r$ ~~just~~ exactly once either barred or unbarred. ~~such that~~ In addition ~~the~~

For spin +: The ~~l~~ letter r ~~appears~~ (resp. $\bar{r}$) appears at height h where $r-h$ is even (resp. odd)

For spin −: The ~~letter~~ r (resp. $\bar{r}$) appears at height h where $r-h$ odd (resp. even) $\Big\}$

$$\mathcal{C}_r^{\text{spin}-} \xrightarrow{\sim} \mathcal{M}_{\varpi_{r-1}}$$

$$\mathcal{C}_r^{\text{spin}+} \xrightarrow{\sim} \mathcal{M}_{\varpi_r}$$

$$\begin{array}{|c|}\hline j_1 \\ \hline \vdots \\ \hline j_r \\ \hline \end{array} \longmapsto \boxed{\epsilon_1 \cdots \epsilon_r}, \quad \epsilon_k = \begin{cases} +, & \text{if } k = j_s \text{ for some } s \\ -, & \text{if } \bar{k} = j_s \text{ for some } s \end{cases}$$

Tableaux of type D_r:

λ dominant, $\lambda = a_1\varpi_1 + \dots + a_r\varpi_r$

$\mathrm{Tab}_\lambda =$ Tableaux T of shape λ in alphabet D_r s.t.

D1) Each column of width 1 height $1, \dots, r-1$ is in $\mathcal{C}_k$.

Each column of width 1 and height r ~~is~~ is in

$$\begin{cases} \mathcal{C}_r^-, & \text{if } a_{r-1} > a_r \\ \mathcal{C}_r^+, & \text{if } a_{r-1} < a_r \end{cases}$$

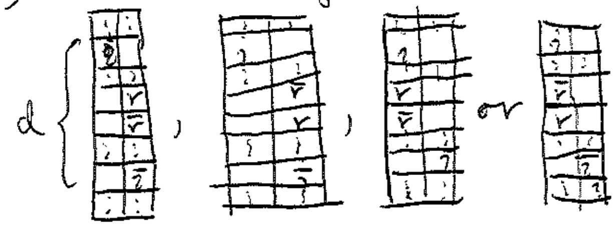
Each column of width and height $\frac{1}{2}$ is in

$$\begin{cases} \mathcal{C}_r^{\text{spin}-}, & \text{if } |a_{r-1} - a_r| \text{ is odd} \\ \mathcal{C}_r^{\text{spin}+}, & \text{if } |a_{r-1} - a_r| \text{ is even} \end{cases}$$

D2) Each row in T is weakly increasing (so r and $\bar{r}$ cannot both appear)

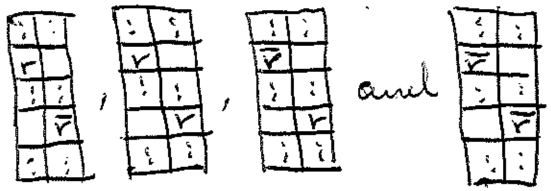
D3) Condition (C3) holds for $1 \le i \le j < r$.

D4) If T has two adjacent columns of the form



~~then~~ where $i \le r$ and $d > 0$, then $d - 1 < r - i$

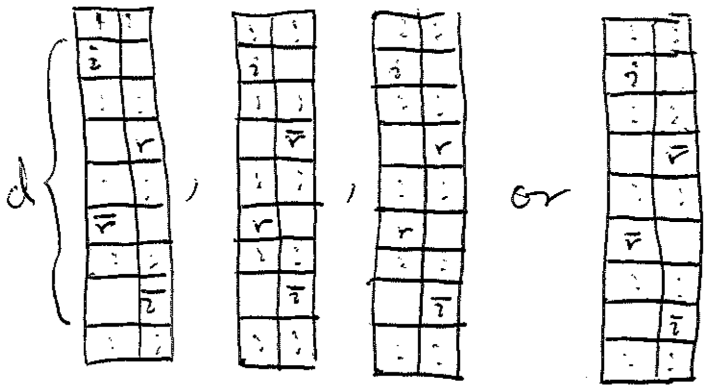
D5) T cannot have adjacent columns of the form



where the two entries are in different rows.

D6) Next page:

D6) If ~~the~~ T has two adjacent columns of the form:



where $i<r$, $d>0$ ~~and~~ is the vertical distance between i and $\bar{i}$, and where the vertical distance between

$$\begin{cases} r \text{ and } \bar{r} \text{ is odd} & \leftarrow \text{first two cases} \\ r \text{ and } r \text{ is even} \\ \bar{r} \text{ and } \bar{r} \text{ is even,} \end{cases} \Big\} \text{ last two cases}$$

then we must have $d < r-i$.

Theorem:

$$Tab_\lambda \cong B_\lambda$$

Example: D6

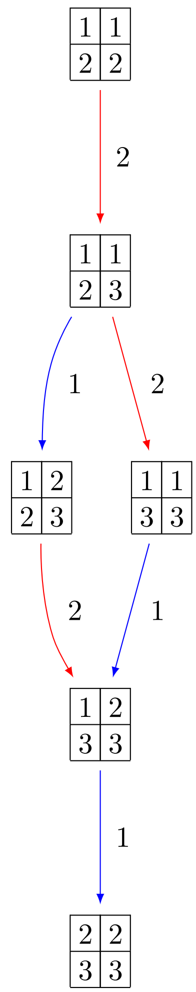
$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 6 \\ \hline \bar{6} & \bar{5} \\ \hline \bar{3} & \bar{1} \\ \hline \end{array} \in Tab_{(2,2,2,2)} \qquad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & 6 \\ \hline \bar{6} & \bar{5} \\ \hline \bar{3} & \bar{2} \\ \hline \end{array} \notin Tab_{(2,2,2,2)}$$

$\hookrightarrow$ D6 is violated

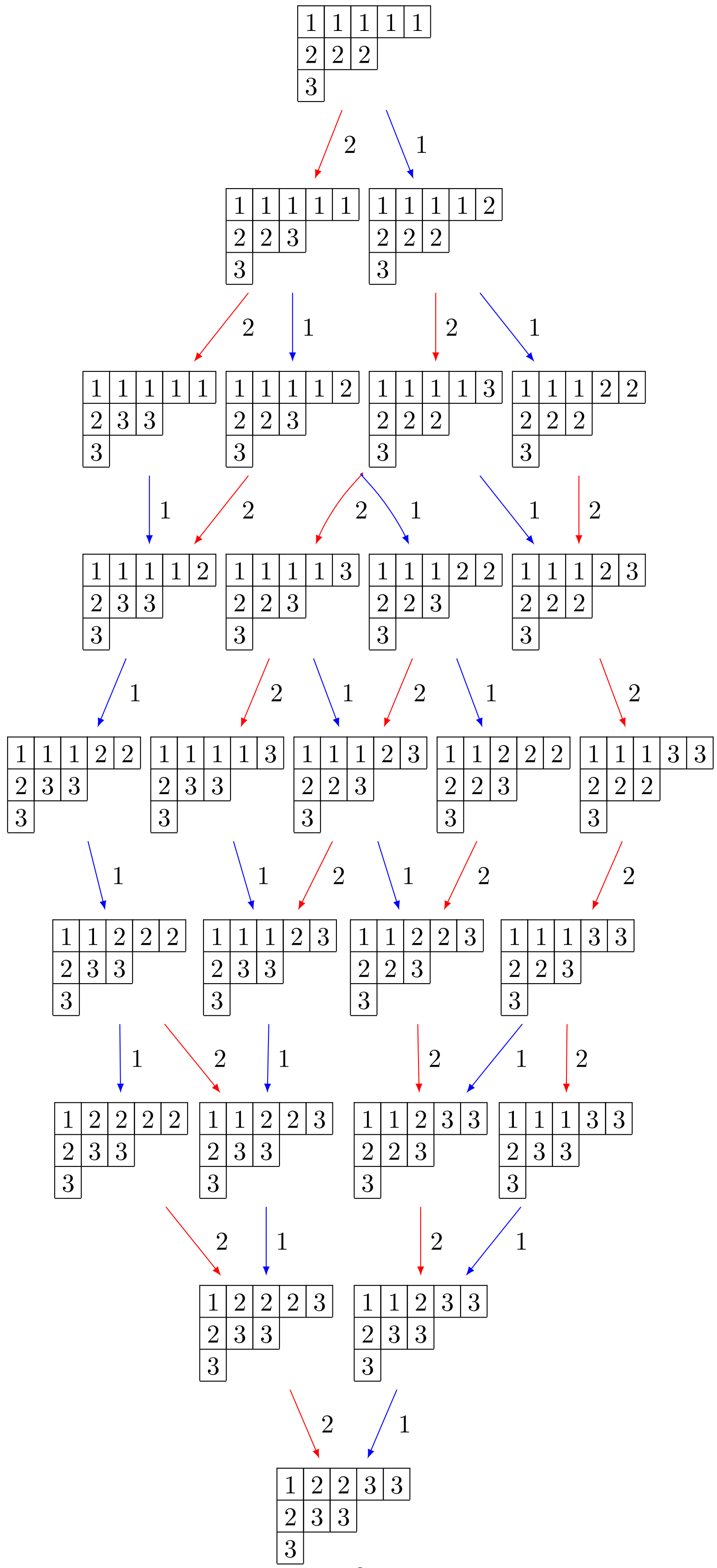
$i=2$, $r=6$, $d=4 \not< r-i=6-2=4$

1 Type A_r

1.1 Example: A_2 and $\lambda = (2, 2, 0) = 2\overline{\omega}_2$

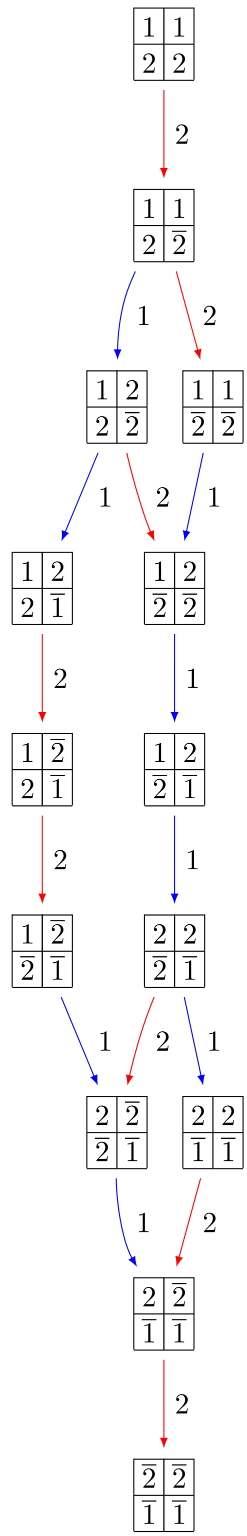


1.2 Example: A_2 and $\lambda = (5,3,1) = 2\overline{\omega}_1 + 2\overline{\omega}_1 + (1,1,1)$



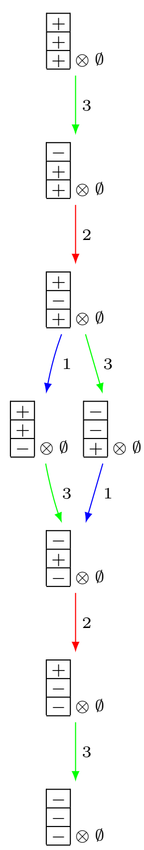
2 Type C_r

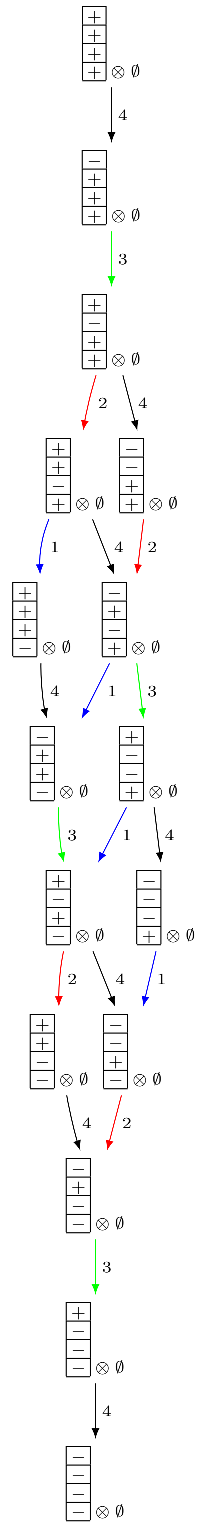
2.1 Example: C_2 and $\lambda = (2,2) = 2\overline{\omega}_2$



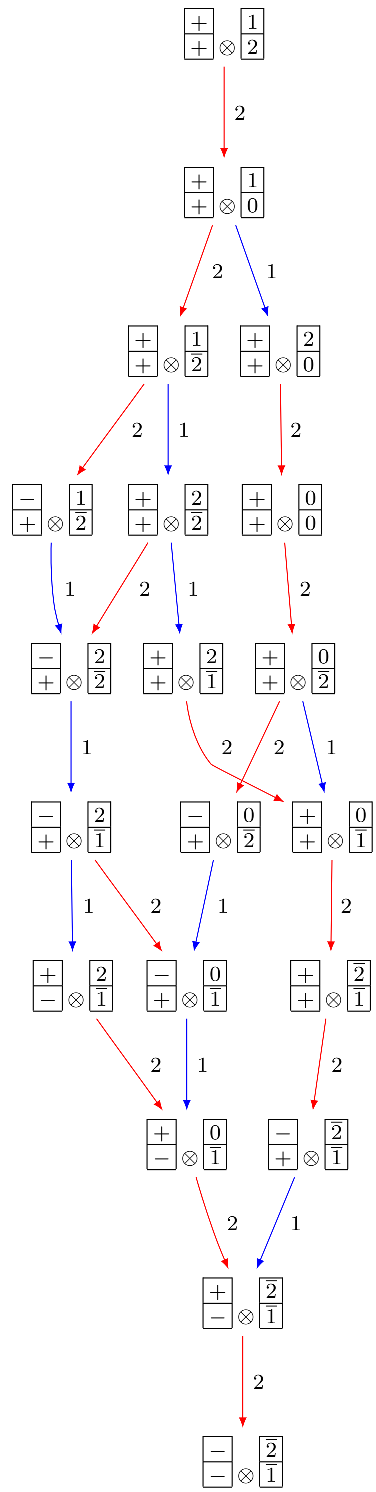
3 Type B_r

3.1 Example: $\mathcal{C}_3^{spin}$ for B_3 and $\mathcal{C}_4^{spin}$ for B_4



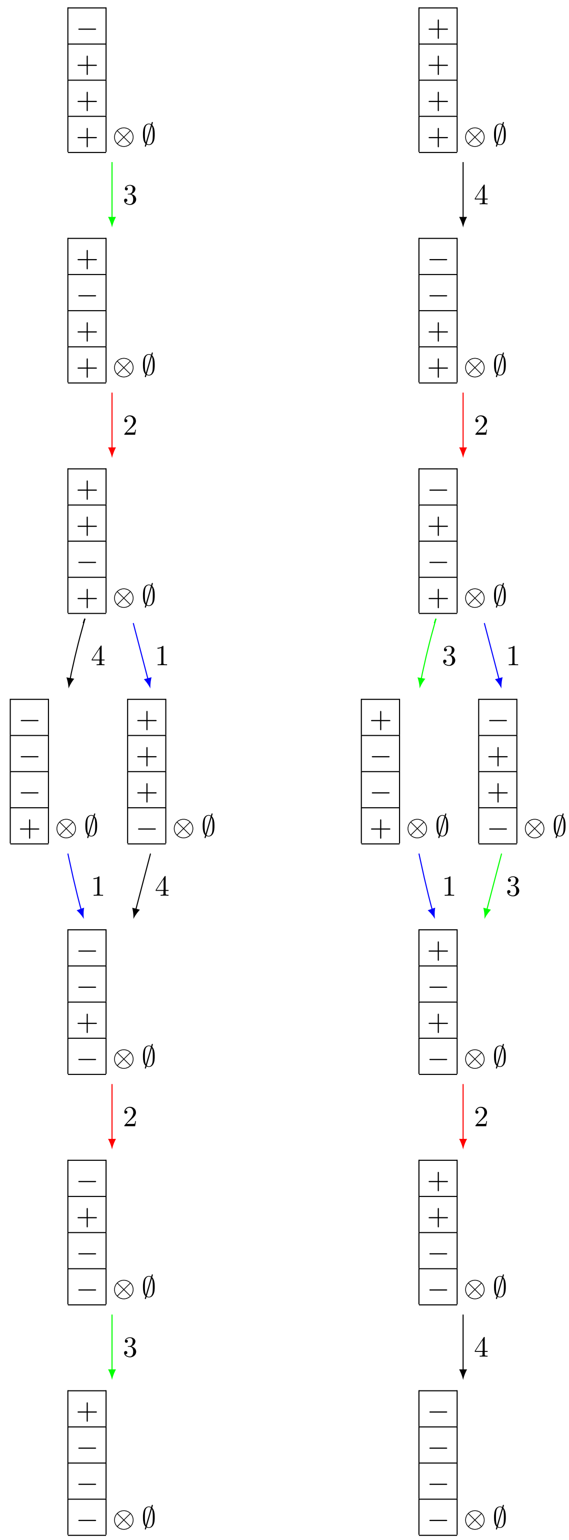


3.2 Example: B_2 and $\lambda = (3/2, 3/2) = 3\overline{\omega}_2$



4 Type D_r:

4.1 Example: $\mathcal{C}_4^{spin-}$ and $\mathcal{C}_4^{spin+}$ for B_4



4.2 Example: D_2 and $\lambda = (3/2, 3/2) = 3\overline{\omega}_2$

$$\begin{array}{c}
\begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \\
\Big\downarrow {\scriptstyle 2} \\
\begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \overline{2} \\ \hline 2 \\ \hline\end{array} \\
\Big\downarrow {\scriptstyle 2} \\
\begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \overline{2} \\ \hline \overline{1} \\ \hline\end{array} \\
\Big\downarrow {\scriptstyle 2} \\
\begin{array}{|c|}\hline - \\ \hline - \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \overline{2} \\ \hline \overline{1} \\ \hline\end{array}
\end{array}$$

4.3 Example: D_2 and $\lambda = (3/2, 3/2) = 3\overline{\omega}_1$

$$\begin{array}{c}
\begin{array}{|c|}\hline - \\ \hline + \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline \overline{2} \\ \hline\end{array} \\
\Big\downarrow {\scriptstyle 1} \\
\begin{array}{|c|}\hline - \\ \hline + \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array} \\
\Big\downarrow {\scriptstyle 1} \\
\begin{array}{|c|}\hline - \\ \hline + \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 2 \\ \hline \overline{1} \\ \hline\end{array} \\
\Big\downarrow {\scriptstyle 1} \\
\begin{array}{|c|}\hline + \\ \hline - \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 2 \\ \hline \overline{1} \\ \hline\end{array}
\end{array}$$

4.4 Example: D_2 and $\lambda = (5/2, 1/2) = 2\overline{\omega}_1 + 3\overline{\omega}_1$