Chapter 4: Stembridge Crystals

Recall:

(1) Fix $\Phi = (\Phi, \Sigma, I, \Lambda)$

$$I \leftrightarrow \Sigma \subset \Phi \subset \Lambda \subset V$$

(2) $\mathcal{C} = (\mathcal{C}, \{e_i, f_i, \varepsilon_i, \varphi_i\}_{i \in I}, wt)$ is a Φ-crystal

(A1) $\forall x, y \in \mathcal{C}$, $y = e_i x$ iff $x = f_i y$

$$\cdots \overset{x}{\bullet} \xleftarrow{i} \overset{y}{\bullet} \cdots \qquad \left\{ \begin{array}{l} wt(y) = wt(x) + \alpha_i \\ \varphi_i(y) = \varphi_i(x) + 1 \\ \varepsilon_i(y) = \varepsilon_i(x) - 1 \end{array} \right.$$

(A2) $\varphi_i(x) - \varepsilon_i(x) = \langle wt(x), \alpha_i^\vee \rangle$

(3) Given $x \in \mathcal{C}$ let

(i) $F_i(x) = \max\{k \geq 0 \,;\, e_i^k(x) \neq 0\}$

(ii) $E_i(x) = \max\{k \geq 0 \,,\, f_i^k(x) \neq 0\}$

If $\{\varepsilon_i, \varphi_i\}$ have (i), (ii) $\Rightarrow$ seminormal

if ε_i has (i) $\Rightarrow$ upper seminormal

(4) $\mathcal{C}, \mathcal{D}$ Φ- crystals

$$\mathcal{C} \otimes \mathcal{D} = (\mathcal{C} \times \mathcal{D}, \{\tilde{e}_i, \tilde{f}_i, \tilde{\varepsilon}_i, \tilde{\varphi}_i\}_{j \in I}, \tilde{wt})$$

$$\tilde{wt}(x \otimes y) = wt(x) + wt(y)$$

$$\tilde{f}_i(x \otimes y) = \begin{cases} f_i x \otimes y & \varepsilon_i(x) \geqslant \varphi_i(y) \\ x \otimes f_i y & \varepsilon_i(x) < \varphi_i(y) \end{cases}$$

$$\tilde{e}_i(x \otimes y) = \begin{cases} e_i x \otimes y & \varepsilon_i(x) > \varphi_i(y) \\ x \otimes e_i y & \varepsilon_i(x) \leqslant \varphi_i(y) \end{cases}$$

$$\tilde{\varphi}_i(x \otimes y) = \max(\varphi_i(x), \varphi_i(y) + \langle wt(x), \alpha_i^\vee \rangle)$$

$$\tilde{\varepsilon}_i(x \otimes y) = \max(\varepsilon_i(y), \varepsilon_i(x) - \langle wt(y), \alpha_i^\vee \rangle)$$

(see § 2.3)

<u>Prop 2.29</u> $\mathcal{C}, \mathcal{D}$ seminormal $\Rightarrow$ $\mathcal{C} \otimes \mathcal{D}$ seminormal

<u>Prop 2.32</u> $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} \longrightarrow \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$ is an iso.

§4.1 Motivation

Fix Φ, let $\mathcal{C}$ be a Φ-crystal.

Def: • $u \in \mathcal{C}$ s.t. $e_i u = 0 \quad \forall i$ is a HWE.

• $x, y \in \mathcal{C}$, set $x \succeq y$ iff $\exists (i_1, \dots, i_k) \in I^k$, $\exists k$ s.t.

$$x = e_{i_1} \cdots e_{i_k}(y)$$

Rem: $x = e_{i_1} \cdots e_{i_k}(y) \Rightarrow wt(x) = wt(y) + \sum_{j=1}^{k} \alpha_{i_j}$

$\Rightarrow wt(x) \succeq wt(y)$.

Lem 4.1 If $\exists!$ HWE $u \in \mathcal{C}$

$$\Longrightarrow \begin{cases} wt(u) \succeq wt(x) \ \forall x \\ \mathcal{C} \text{ connected} \end{cases}$$

Pf: Let $M = \{ y \in \mathcal{C} ; y \succeq x, \forall x \in \mathcal{C} \}$

$\Rightarrow M \neq \emptyset$, since o.w. $(\forall y, \exists x \text{ s.t. } y \prec x)$ $\exists x$ s.t. $u \prec x$.

$\Rightarrow M = \{u\}$. $\square$

Problem 4.2 Can we define a map

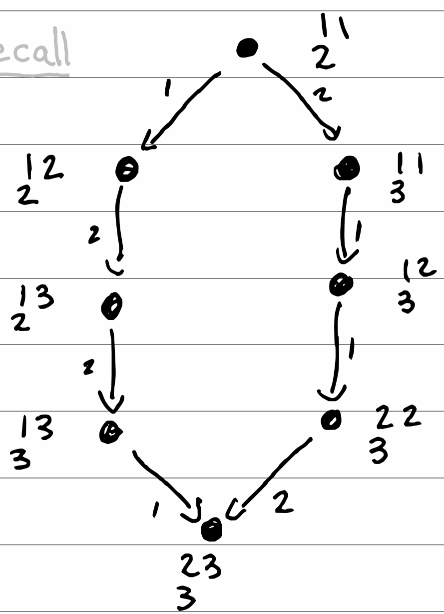
$$\Lambda^+ \longrightarrow \left\{ \begin{array}{l} \text{semi-normal } \Phi\text{-crystals} \\ \exists!\, u_\lambda \text{ HWE},\ wt(u_\lambda) = \lambda \end{array} \right\}$$

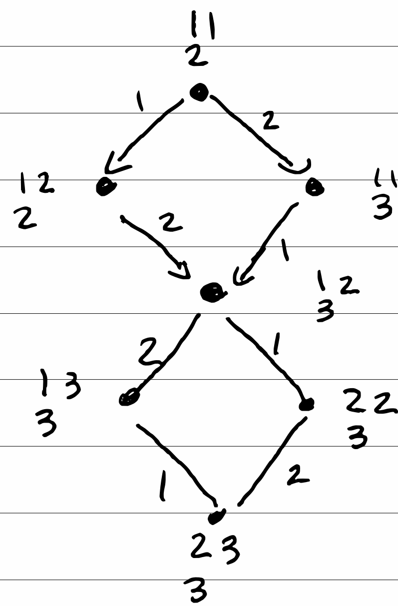
$$\lambda \longmapsto B_\lambda .$$

which is closed under $\otimes$?

(also: $\mathrm{char}(B_\lambda) = \mathrm{char}(V_\lambda)$)

Recall





f_i: rightmost $i \mapsto i+1$

For Φ simply-laced: Stembridge!

Given $\Phi = (\Phi, \Sigma, I, \Delta)$

$\Downarrow$ $J \subseteq I$

$\Phi_J = (\Phi_J, \Sigma_J, J, \Delta)$

Levi Branching.

If $\mathcal{C}$ is a Φ-crystal

$\Rightarrow$ $\mathcal{C}$ is also a Φ_J-cristal (disregard $\{e_i, f_i, \varepsilon_i, \varphi_i\}$ $i \notin J$).

Idea: Consider branchings for $J \subseteq I$ with $|J| = 2$

Possibilities:

Φ_J of type $\begin{cases} A_1 \times A_1 \text{ or} \\ A_2 \end{cases}$

if Φ is simply-laced.

Lem 4.4 Assume $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ & $e_i x \neq 0$. Then

$$\varphi_j(e_i x) - \varphi_j(x) = \varepsilon_j(e_i x) - \varepsilon_j(x) - 1$$

Pf: (A1): $\langle wt(e_i x), \alpha_j^\vee \rangle = \langle wt(x), \alpha_j^\vee \rangle - 1$

(A2): $\varphi_j(e_i x) - \varepsilon_j(e_i x) = \varphi_j(x) - \varepsilon_j(x) - 1$ $\square$

§4.2 Stembridge Axioms.

Let $\mathcal{C}$ be a Φ-crystal with Φ simply-laced.

(S0) $e_i x = 0 \Rightarrow \varepsilon_i x = 0$, $\forall i$

(S0′) $f_i x = 0 \Rightarrow \varphi_i x = 0$

Rem: $\mathcal{C}$ seminormal $\Rightarrow$ (S0), (S0′).

(S1) $i \neq j \in I$, $x, e_i x \in \mathcal{C}$..

$$\Rightarrow \varepsilon_j(e_i x) = \begin{cases} \varepsilon_j(x) \\ \varepsilon_j(x) + 1 \iff \langle \alpha_i, \alpha_j^\vee \rangle = -1. \end{cases}$$

Prop 4.5 $\mathcal{C}$ satisfies (S1) & $e_i x \neq 0$.
Then, we have 3 possibilities

(1) $\varepsilon_j(e_i x) = \varepsilon_j(x)$ $\quad\varphi_j(e_i x) = \varphi_j(x) - 1$ $\Big\}$ $\langle \alpha_i, \alpha_j^\vee \rangle = -1$

(2) $\varepsilon_j(e_i x) = \varepsilon_j(x) + 1$ $\quad\varphi_j(e_i x) = \varphi_j(x)$

(3) $\varepsilon_j(e_i x) = \varepsilon_j(x)$ $\quad\varphi_j(e_i x) = \varphi_j(x)$ $\big\{$ $\alpha_i \perp \alpha_j$

Pf: $\varphi_j(e_i x) - \varepsilon_j(e_i x) = \langle \mathrm{wt}(e_i x), \alpha_j^\vee \rangle$ (A2)

$= \langle \mathrm{wt}(x), \alpha_j^\vee \rangle + \langle \alpha_i, \alpha_j^\vee \rangle$ (A1)

$= \varphi_j(x) - \varepsilon_j(x) + \langle \alpha_i, \alpha_j^\vee \rangle$. (A2)

Claim follows from (S1). $\blacksquare$

(S2) $i \neq j \in I$, $x \in \mathcal{C}$ with

$$\begin{cases} \varepsilon_i(x) > 0 & \& \\ \varepsilon_j(e_i x) = \varepsilon_j(x) > 0 \end{cases}$$

$$\Rightarrow \begin{cases} e_i e_j x = e_j e_i x & \& \\ \varphi_i(e_j x) = \varphi_i(x) \end{cases}$$

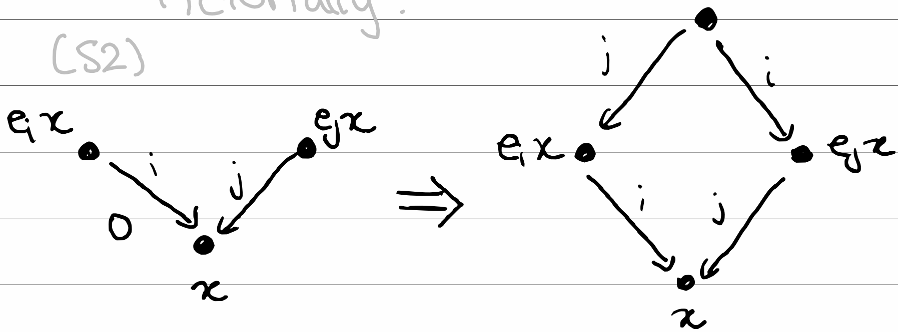
(S3) $i \neq j \in I$, $x \in \mathcal{C}$ with

$$\begin{cases} \varepsilon_j(e_i x) = \varepsilon_j(x) + 1 > 1 \\ \varepsilon_i(e_j x) = \varepsilon_i(x) + 1 > 1 \end{cases}$$

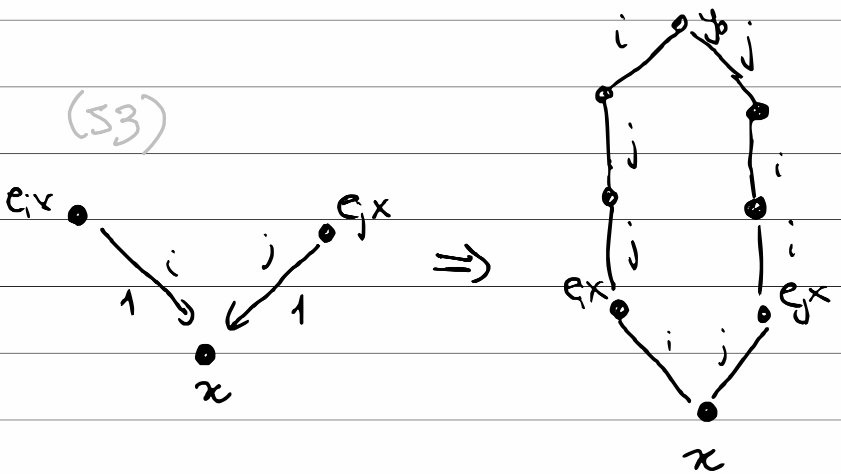
$$\Rightarrow \begin{cases} e_j e_i^2 e_j x = e_i e_j^2 e_i x, \\ \varphi_i(e_j x) = \varphi_i(e_j^2 e_i x) \\ \varphi_j(e_i x) = \varphi_j(e_i^2 e_j x). \end{cases}$$

Pictorially:

(S2)



(S3)



Lem 4.6 e satisfying (S0–S3), x, i, j as in (S3). Then, all 8 elements are distinct.

Pf: From wt possibly $e_ie_jx = e_je_ix$.

Say $e_ie_jx = e_je_ix$ (CA)

$$\Rightarrow \varepsilon_j(e_ie_jx) = \varepsilon_j(e_je_ix)$$

$$= \varepsilon_j(e_ix) - 1 \qquad (\cancel{A2}\ A1)$$

(*) $$= \varepsilon_j(x) \qquad (S3)$$

$$= \varepsilon_j(e_jx) + 1$$

Let $z = e_j x$.

$$(*) \implies \varepsilon_j(e_i z) = \varepsilon_j(z) + 1$$

$$(4.5)_{ii} \implies \varphi_j(e_i z) = \varphi_j(z)$$

$$\iff \varphi_j(e_i e_j x) = \varphi_j(x) + 1$$

Now:

$$\begin{aligned}
\varphi_j(e_i^2 e_j x) &= \varphi_j(e_i x) && (S3)\\
&= \varphi_j(x) && (*) + (4.5)\\
&= \varphi_j(e_j e_i x) - 1 && (\cancel{A2}\ A1)\\
(\dagger) \qquad &= \varphi_j(e_i e_j x) - 1 && (CA)
\end{aligned}$$

$$(4.5)_{(i)} \implies \varepsilon_j(e_i^2 e_j x) = \varepsilon_j(e_i e_j x)$$

$$(S2) \implies \varphi_i(e_j e_i e_j x) = \varphi_i(e_i e_j x)$$

$$(CA) \iff \varphi_i(e_j^2 e_i x) = \varphi_i(e_j e_i x)$$

Swapping i,j above

$$\varphi_j(e_i^2 e_j x) = \varphi_j(e_i e_j x)$$

contradicting $(\dagger)$ $\square$

Dual axioms:

(S1′) $i \neq j \in I$, $x, f_i x \in \mathcal{C}$

$$\Rightarrow \varphi_j(f_i x) = \begin{cases} \varphi_j(x) \\ \varphi_j(x) + 1 \iff \langle \alpha_i, \alpha_j^\vee \rangle = -1 \end{cases}$$

(S2′) $i \neq j \in I$, $x, f_i x \in \mathcal{C}$ with

$$\begin{cases} \varphi_i(x) > 0 \\ \varphi_j(f_i x) = \varphi_j(x) > 0 \end{cases}$$

$$\Rightarrow \begin{cases} f_i f_j x = f_j f_i x \\ \varepsilon_i(f_j x) = \varepsilon_i(x) \end{cases}$$

(S3′) $i \neq j \in I$, $x \in \mathcal{C}$ with

$$\begin{cases} \varphi_j(f_i x) = \varphi_j(x) + 1 > 1 \\ \varphi_i(f_j x) = \varphi_i(x) + 1 > 1 \end{cases}$$

$$\Rightarrow \begin{cases} f_j f_i^2 f_j x = f_i f_j^2 f_i x \neq 0 \\ \varepsilon_i(f_j x) = \varepsilon_i(f_j^2 f_i x) \\ \varepsilon_j(f_i x) = \varepsilon_j(f_i^2 f_j x) \end{cases}$$

Def: • $\mathcal{C}$ weakly Stembrige if (S0 – S3) & (S0′ – S3′) holds
• $\mathcal{C}$ Stembrige if weakly Stembridge + seminormal.

Prop 4.7 $\mathcal{C}$ wSt,
$i \neq j$ with $\langle \alpha_i, \alpha_j^\vee \rangle \overset{\vee}{=} -1$
If $x \in \mathcal{C}$, $\begin{cases} \varepsilon_j(x) > 0 \\ \varepsilon_i(e_j x) = \varepsilon_i(x) + 1 \end{cases}$
$\Rightarrow \varepsilon_j(e_i e_j x) = \varepsilon_j(x) - 1$

Rem 4.8 $\mathcal{C}$ fin. type. $\mathcal{C}$ satisfies (S1 – S3) iff $\mathcal{C}^\vee$ satisfies (S1′ – S3′)

Prop 4.9 $\mathcal{C}$ wSt.
$x \in \mathcal{C}$ with $\begin{cases} e_j x, e_i x, e_j e_i x, e_j^2 e_i x \neq 0 \\ \varphi_i(e_j x) < \varphi_i(x) \end{cases}$
$\Rightarrow \varphi_i(e_j^2 e_i x) < \varphi_i(e_j e_i x)$.

§4.3 Stembridge Crystals are mon. cat.

Thm 4.10 $\mathcal{C}, \mathcal{D}$ St Cry. $\Rightarrow$ $\mathcal{C} \otimes \mathcal{D}$ St Cry

Sketch (S0, S0′) follow from semi-normal.

(S1 – S3): lots of cases! (omit)

...

Now: (S2′), (S3′) hold for $\mathcal{C}, \mathcal{D}$ (seminormal & fin. type)

$\Rightarrow$ (S2), (S3) hold for $\mathcal{C}^\vee, \mathcal{D}^\vee$

$\Rightarrow$ (S2), (S3) hold for $\mathcal{D}^\vee \otimes \mathcal{C}^\vee = (\mathcal{C} \otimes \mathcal{D})^\vee$

$\Rightarrow$ (S2′), (S3′) hold for $(\mathcal{C} \otimes \mathcal{D})$ $\square$

§4.4 Properties of St Cry.

Thm 4.11 $\mathbb{B}$ the std A_r or D_r crystal.

Any full (i.e. union of components) subcrystal of $\mathbb{B}^{\otimes k}$ is St Cry

In particular any Cryst. of tableaux is St Cry.

Sketch: $\mathbb{B}(A_r)$: $\textcircled{1} \xrightarrow{1} \textcircled{2} \xrightarrow{2} \textcircled{3} \xrightarrow{3} \cdots \xrightarrow{r-1} \textcircled{r} \xrightarrow{r} \textcircled{r+1}$

$\mathbb{B}(D_r)$:

$$\begin{tikzcd}[column sep=small] & & & & r \arrow[dr, "r"] & & & & \\ 1 \arrow[r, "1"] & 2 \arrow[r, "2"] & \cdots \arrow[r, "r-2"] & r-1 \arrow[ur, "r-1"] \arrow[dr, "r"'] & & \overline{r-1} \arrow[r, "r-2"] & \cdots \arrow[r, "1"] & \bar{1} \\ & & & & \bar{r} \arrow[ur, "r-1"'] & & & \end{tikzcd}$$

Both are checked to be StCry $\square$

Thm 4.2 [St, 03]

$\mathcal{C}$ ω StCry, non-empty, upper seminormal and bounded above (i.e. $\forall x$, $\exists$ HWE y s.t. $x \preceq y$)
$\Rightarrow$ $\mathcal{C}$ has a unique HWE.

Pf: $\mathcal{C} \neq \emptyset$ bdd above $\Rightarrow$ $\exists$ x max'l element.
(any HWE)

$\Omega := \{ y ;\ y \preceq x \}$

$S := \{ y ;\ y \in \Omega$ but $0 \neq e_i y \notin \Omega, \exists i \}$

Claim: $S = \emptyset$.

If not, let $y \in S$ be maximal (Zorn Lemma)

$\left.\begin{array}{l} x \succ y \\ y \in S \end{array}\right\} \Rightarrow \exists\ i \neq j \in I$ s.t. $e_i y \neq 0$, $e_j y \neq 0$.

$\Rightarrow \begin{cases} (S2) \Rightarrow e_i e_j y = e_j e_i x \neq 0 & \text{or} \\ (S3) \Rightarrow e_i e_j^2 e_i y = e_j e_i^2 e_j y \neq 0 . \end{cases}$

$$e_j y \succ y \Rightarrow e_j y \notin S$$

$$\Rightarrow e_i e_j y \in \Omega \overset{(S2)}{\Longrightarrow} e_i y \preceq e_j e_i y \preceq x \ ↯$$

$$\overset{(S3)}{\Longrightarrow} e_i e_j y \notin S$$

$$\Rightarrow e_i^2 e_j y \in \Omega \setminus S$$

$$\Rightarrow e_j e_i^2 e_j y = e_i e_j^2 (e_i y) \in \Omega \ ↯$$

So $S = \emptyset$.

<u>Claim:</u> $\mathcal{C} = \Omega$

$$\left.\begin{array}{l} \mathcal{C} \text{ conn.} \\ \Omega \neq \emptyset \end{array}\right\} \Rightarrow \exists y \in \Omega \text{ s.t. } \left\{\begin{array}{l} e_i y \notin \Omega \Rightarrow y \in S = \emptyset \ ↯ \\ f_i y \notin \Omega \Rightarrow f_i y \prec y \preceq z \\ \quad \Rightarrow f_i y \in \Omega \ ↯ \end{array}\right.$$

<u>Thm 4.13 [St 03]</u>

$\mathcal{C}, \mathcal{C}'$ connected St Cry, $u \in \mathcal{C}$, $u' \in \mathcal{C}'$ HWE.

If $wt(u) = wt(u') \Rightarrow \mathcal{C} \simeq \mathcal{C}'$