Chapter 2: Kashiwara crystals

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EOS
THE EXCELLENCE OF SCIENCE

Outline

1. 2.1 Root systems
2. 2.2 Kashiwara crystals
3. 2.3 Tensor products of crystals
4. 2.4 The signature rule ok
5. 2.5 Root strings
6. 2.6 The character
7. 2.7 and 2.8: skipped

2.1 Root systems

Root systems

Denote V a Euclidean space with inner product $\langle -, - \rangle$. For $\alpha \in V$

co-vector

$$r_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad \alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$

$$= x - 2\frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

$$r_x(x) = x - 2\frac{\langle x, x \rangle}{\langle x, x \rangle}x = -x$$

ex: $\mathbb{R}$ (1d)

Definition

A root system $\Phi \subset V$ is a finite set of vectors such that $\alpha, \beta \in \Phi$

1. $\alpha \neq 0$
2. $r_\alpha(\Phi) = \Phi$, $\alpha, \beta \in \Phi$ $r_\alpha(\beta) \in \Phi$
3. $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ crystallographic
4. if $\beta = c\alpha$ then $c = \pm 1$. reduced

All crystallographic root systems in 2D

positive

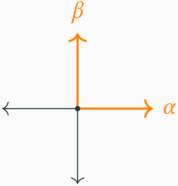
negative

$\alpha+\beta$

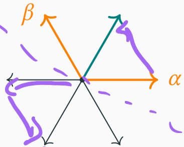
simply-laced

reducible

$A_1 \oplus A_1$:



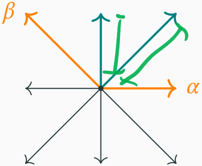
A_2 :



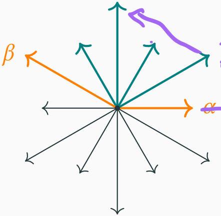
$-\alpha-\beta$

$2\alpha+\beta$

B_2 :



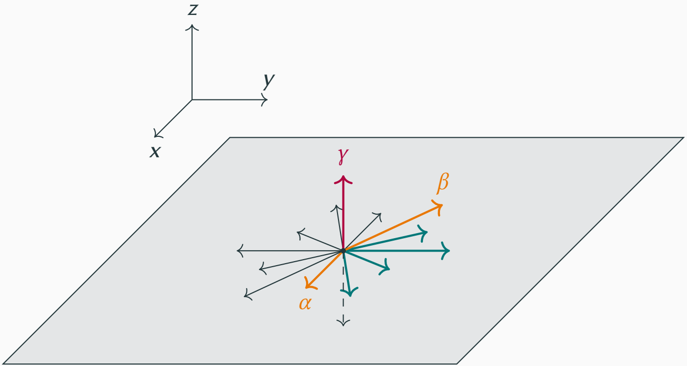
G_2 :



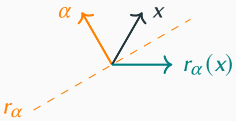
irreducible

A reducible root system in 3D

$G_2 \oplus A_1$



$$r_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha$$



Weyl group

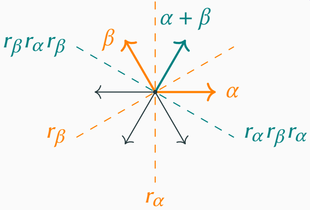
The reflection group $W(\Phi) = \langle r_\alpha \mid \alpha \in \underline{\Phi^+}\rangle \subset \mathcal{O}(N)$ is the Coxeter group of the root system.

$W = \langle r_\alpha \mid r_\alpha \text{ simple roots} \rangle$

$\simeq \langle r_\alpha \mid r_\alpha^2 = 1,\ (r_\alpha r_\beta)^{m_{\alpha\beta}} = 1 \rangle$

$m_{\alpha\beta} \in \mathbb{N}$

$\Phi := A_2 :$



$W(\Phi) = \mathbb{S}_3 \simeq D_6.$

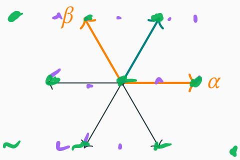
$r_\alpha \quad r_\beta$

$r_\alpha r_\beta r_\alpha = r_\beta r_\alpha r_\beta \qquad (r_\alpha r_\beta)^3 = 1$

Weight lattice

GL_3

A_2 $\Lambda = \mathbb{Z}^2$



Definition

$\Lambda \subset V$ is a lattice if $\lambda \in \Lambda$ is a weight

1. it spans V
2. $\Phi \subset \Lambda$
3. if $\lambda \in \Lambda$ and $\alpha \in \Phi$ then $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$

semisimple: if Φ spans Λ

$\Lambda \supset \Lambda_\Phi$

Order

I : index set of the simple roots $\subset \Phi^+$

Definition

A partial order $\succcurlyeq$ is defined on Λ by $\lambda \succcurlyeq \mu$ if

$$\lambda - \mu = \sum_{i \in I} c_i \alpha_i, \quad c_i \geq 0$$

$\Lambda^+ : \{ \lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ \forall i \in I \}$ $>$

Dominant weight

strictly

The fundamental weights ϖ_i — simple root

$$\left\langle \varpi_i, \alpha_j^\vee \right\rangle = \delta_{ij}. \quad i, j \in I$$

$\Lambda_{\Delta} \supseteq \Lambda \supseteq \Lambda_\Phi$

lattice generated by fundamental weight

Example: $\mathrm{GL}(r+1)$

$A_r \rightsquigarrow (r+1)$ type

$\Phi : \{e_i - e_j \mid i \neq j\}$ $\qquad e_i$: unit vector

$\Phi^+ : \{e_i - e_j \mid i < j\}$,

$\Lambda = \mathbb{Z}^{r+1}$

$\lambda = (\lambda_1 \cdots \lambda_{r+1})$ is dominant if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+1}$

simple roots $e_1 - e_2 \cdots e_r - e_{r+1}$

fundamental weights are

$$\varpi_i = (\underbrace{1 \cdots 1}_{i}, 0 \cdots 0)$$

2.2 Kashiwara crystals

Crystals

Crystals

$\Phi + \Lambda$

Definition

A Kashiwara crystal of type Φ is a set $\mathcal{B}$ with maps $i \in I$

\# elements degree

$$e_i, f_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$$

crystal operation $0 \notin \mathcal{B}$

$$\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$$

string lenght

$$\mathrm{wt} : \mathcal{B} \to \Lambda$$

weight

respecting

$$x \underset{f_i}{\overset{e_i}{\rightleftarrows}} y$$

A1 $e_i(x) = y$ iff $f_i(y) = x$ and then

$$\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i,\ \ \varepsilon_i(y) = \varepsilon_i(x) - 1,\ \ \varphi_i(y) = \varphi_i(x) + 1$$

A2 $\varphi_i(x) = \langle \mathrm{wt}(x), \alpha_i^\vee \rangle + \varepsilon_i(x)$

$\to$ $\varphi_i(x) = -\infty$ then $\varepsilon_i(x) = -\infty$ and then $e_i(x) = f_i(x) = 0$

Seminormal

Definition

If $-\infty$ is not in the image of ε_i, φ_i, then $\mathcal{B}$ is of finite type.

It is seminormal if

$$\varphi_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k(x) \neq 0\}$$
$$\varepsilon_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} \mid e_i^k(x) \neq 0\}$$

⚠ $\mathcal{B}$ finite type $\not\Rightarrow$ finite

Seminormal $\Rightarrow$ finite type

One proposition

Proposition

Φ semisimple and C crystal of finite type.

$$\mathrm{wt}(x) = \sum_{i \in I} (\varphi_i(x) - \varepsilon_i(x))\varpi_i.$$

Crystal graph

$f_i(x)=y$

$x \xrightarrow{i} y \xrightarrow{2} z$

Definition

Construct a quiver from $\mathcal{B}$ by drawing an edge $x \xrightarrow{i} y$ if $f_i(x) = y$.

Can have an equivalence relation on $\mathcal{B}$ if two elements are linked. The equivalence classes are connected components of the graph.

Some propositions on highest weight

Highest weight

An element $u \in \mathcal{B}$ such that $e_i(u) = 0, \forall i \in I$ is called highest weight element and $\mathrm{wt}(u)$ a highest weight.

Lemma

If $\mathrm{wt}(u)$ is maximal with respect to $\succcurlyeq$ then u is a highest weight element.

Proposition

$\mathcal{B}$ seminormal. If u highest weight element, then $\mathrm{wt}(u)$ is dominant.

Examples: A_r and dual crystal

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

$\mathcal{B}: \{\boxed{1}, \boxed{2}, \boxed{3}\}$

$e_i : \boxed{i+1} \mapsto \boxed{i}$

$f_i : \boxed{i} \mapsto \boxed{i+1}$

if not a box $\mapsto 0$

$wt : \boxed{i} \mapsto e_i$

Assume seminormal

$\varepsilon_i = \max\{k \mid e_i^k x \neq 0\}$

$\varphi_i = \max\{k \mid f_i^k x \neq 0\}$

$\varepsilon_1 \; \boxed{1} = 0$

$\boxed{2} = 1$

$\boxed{3} = 0$

$\varphi_1 \; \boxed{1} = 1$

$\boxed{2} = 0$

$\boxed{3} = 0$

$\mathcal{B}^\vee$: dual

$$\boxed{\bar{3}} \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

$wt(\boxed{\bar{j}}) = -e_j$

Example: crystal of rows

row λ_1. $\Phi = A_2$ $\Lambda = \mathbb{Z}^3$

rows in Young tableau

$\boxed{j_1 | j_2 | j_3}$

$B_{(3)}$ = set of rows of length 3

φ_i = # i in $\{j_1\ j_2\ j_3\}$

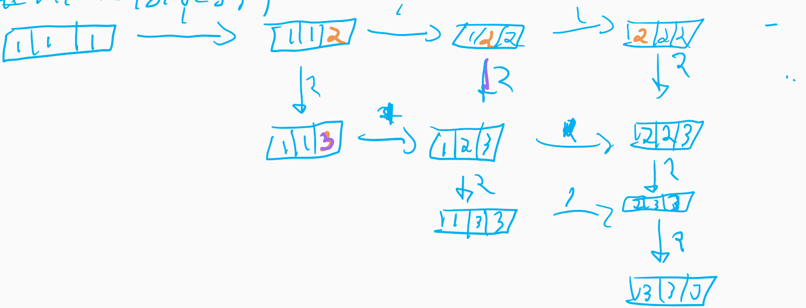
ε_i = # $i+1$ in $\{j_1\ j_2\ j_3\}$

if $\varphi_i > 0$ $f_i R$ = row rightmost $i \mapsto i+1$

if $\varphi_i = 0$ then $f_i = 0$

if $\varepsilon_i > 0$ $e_i(R)$ = row with leftmost $i+1 \mapsto i$

if $\varepsilon_i = 0$ $e_i R = 0$



2.3 Tensor products of crystals

$\mathcal{B}$ and $\mathcal{C}$ crystal associated to Φ. $\mathcal{B} \otimes \mathcal{C}$ with maps

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \varphi_i(y) \leq \varepsilon_i(x) \\ x \otimes f_i(y) & \varphi_i(x) > \varepsilon_i(x) \end{cases}$$

$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y & \varphi_i(y) < \varepsilon_i(x) \\ x \otimes e_i(y) & \varphi_i(x) \geq \varepsilon_i(x) \end{cases}$$

$$\varphi_i(x \otimes y) = \max(\varphi_i(x), \varphi_i(y) + \langle \mathrm{wt}(x), \alpha_i^\vee \rangle)$$

$$\varepsilon_i(x \otimes y) = \max(\varepsilon_i(y), \varepsilon_i(x) - \langle \mathrm{wt}(y), \alpha_i^\vee \rangle)$$

Proposition

It is a crystal.

A1 $f_i(x \otimes y) = z \otimes w$

$\Updownarrow$

$e_i(z \otimes w) = x \otimes y$

$\Downarrow$

Case 1 $\varphi_i(y) \leq \varepsilon_i(x)$

$f_i(x \otimes y) = f_i(x) \otimes y = z \otimes w$

$e_i(z \otimes w) = e_i(z) \otimes w$

$\mathcal{B}$ A1 $x \otimes y$

Tensor product

$\mathcal{B}$ and $\mathcal{C}$ crystal associated to Φ. $\mathcal{B} \otimes \mathcal{C}$ with maps

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \varphi_i(y) \leq \varepsilon_i(x) \\ x \otimes f_i(y) & \varphi_i(x) > \varepsilon_i(x) \end{cases}$$

$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y & \varphi_i(y) < \varepsilon_i(x) \\ x \otimes e_i(y) & \varphi_i(x) \geq \varepsilon_i(x) \end{cases}$$

$$\varphi_i(x \otimes y) = \mathsf{max}(\varphi_i(x), \varphi_i(y) + \left\langle \mathsf{wt}(x), \alpha_i^\vee \right\rangle)$$

$$\varepsilon_i(x \otimes y) = \mathsf{max}(\varepsilon_i(y), \varepsilon_i(x) - \left\langle \mathsf{wt}(y), \alpha_i^\vee \right\rangle)$$

Example: Two $\mathrm{GL}(3)$ crystals

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \varphi_i\, y \le \varepsilon_i\, x \\ x \otimes f_i(y) & \varphi_i\, y > \varepsilon_i(x) \end{cases}$$

$$\begin{tikzcd}
 & \boxed{1} \otimes \boxed{1} \arrow[d, "1"] & \\
 & \boxed{1} \otimes \boxed{2} \arrow[dl, "1"'] \arrow[dr, "2"] & \\
\boxed{2} \otimes \boxed{2} \arrow[dr, "2"'] & & \boxed{1} \otimes \boxed{3} \arrow[dl, "1"] \\
 & \boxed{2} \otimes \boxed{3} \arrow[d, "2"] & \\
 & \boxed{3} \otimes \boxed{3} &
\end{tikzcd}$$

f_1 $\quad \varphi_1 \boxed{1} \quad \varepsilon_1 \boxed{1}$

$1 > 0$

$f_1 : \quad \varphi_1 \boxed{2} \quad \varepsilon_1 \boxed{1}$

$0 \le 0$

$f_2 \quad \varphi_2 \boxed{2} \quad \varepsilon_2 \boxed{1}$

$1 > 0$

$$\begin{tikzcd}
\boxed{2} \otimes \boxed{1} \arrow[d, "2"] \\
\boxed{3} \otimes \boxed{1} \arrow[d, "1"] \\
\boxed{1} \otimes \boxed{2}
\end{tikzcd}$$

?

$f_1 \quad \varphi_1 \boxed{1}$

$\shortparallel$

1

$\varepsilon_1 \boxed{3}$

$\shortparallel$

1

$1 \le 1$

$f_1 \boxed{3} = 0$

$f_2 : \quad \varphi_2 \boxed{1} = 0$

$\varepsilon_2 \boxed{2} = 0$

$0 \le 0$

Morphisms and monoidal

Definition Crystal morphism

A morphism between two crystals is a map $\psi : \mathcal{B} \to \mathcal{C} \cup \{0\}$ such that

1. $b \in \mathcal{B}, \psi(b) \in \mathcal{C}$ then:
   1.1 $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$
   1.2 $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$
   1.3 $\varphi_i(\psi(b)) = \varphi_i(b)$.
2. $b, e_i b \in \mathcal{B}$ with $\psi(b), \psi(e_i b) \in \mathcal{C}$ then $\psi(e_i b) = e_i \psi(b)$
3. $b, f_i b \in \mathcal{B}$ with $\psi(b), \psi(f_i b) \in \mathcal{C}$ then $\psi(f_i b) = f_i \psi(b)$

it is an iso if

$\mathcal{B} \cup \{0\} \xleftrightarrow{\text{bij}} \mathcal{C} \cup \{0\}$

Tensor product associativity

The set bijection $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} \to \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$ is a crystal isomorphism.

2.4 The signature rule

$\otimes$ A_2 GL_3

f_2 $\boxed{1} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{3}$

$\boxed{2} \to -$

$\boxed{3} \to +$

$\boxed{3}$ $\boxed{3}$

$0 \quad + \quad + \quad - \quad - \quad 0 \quad \boxed{-} \quad +$

$+ \quad -$

$f_2(-)=0$

$\boxed{1} \otimes \boxed{3} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{3}$

Lemma 2.33 + sect 2.4

2.5 Root strings

One defintion

A map

Let $k = \left\langle \mathrm{wt}(x), \alpha_i^\vee \right\rangle$.

$$\sigma_i(x) = \begin{cases} f_i^k(x) & k > 0 \\ x & k = 0 \\ e_i^{-k}(x) & k < 0 \end{cases}$$

$\sigma_i \mathcal{B} = \mathcal{B}$ and $\mathrm{wt}(\sigma_i(x)) = s_i(\mathrm{wt}(x))$.

2.6 The character

The character

$\Lambda = \mathbb{Z}^{r+1}$

$\mathcal{E} \sim$ Laurent poly in $t_1 \cdots t_{r+1}$

ex in A_r for the crystal of rows.

$t^{\mathrm{wt}\,\boxed{j_1 \mid j_2 \mid j_3}} = \prod_i t_{j_i}$

$t^{\mu} = \prod_{i=1}^{r+1} t_i^{\mu_i}$

Let $\mathcal{E}$ be the free abelian group on Λ with basis element $t^{\mu}, \mu \in \Lambda$. The character is

$$\chi_{\mathcal{B}}(t) = \sum_{v \in \mathcal{B}} t^{\mathrm{wt}(v)}.$$

The character is invariant under W.

$\chi_{\mathcal{B}(\omega)} t = \sum_{j_1 \le j_2 \le \cdots \le j_k} t_{j_1} \cdots t_{j_k}$

symmetric polynomial

2.7 and 2.8: skipped

Questions?