Tensor Categories, Sections 9.9–9.10

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- all symm fiber functs are isomorphic
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Plan: 1, 2, 3, 4, 5, 6, 7, 8, 9 + sketch of proofs.

Before the fun

Last time:
$\mathcal{C}$ fusion cat then $\mathcal{C}_{ad}$ smallest tensor Serre subcat containing $X \otimes X^*, X \in \mathcal{O}(\mathcal{C})$.

If G finite group then $\mathrm{Rep}(G)_{\mathrm{ad}} = \mathrm{Rep}(G/Z_G)$:

$$G \curvearrowright V \Rightarrow G \curvearrowright V \otimes V^* \cong \mathrm{End}(V)$$

$$g \cdot T = g \circ T \circ g^{-1}, \quad T \in \mathrm{End}(V)$$

$$\Rightarrow \begin{tikzcd} G \arrow[rr] \arrow[dr] & & \mathrm{Aut}(V \otimes V^*) \\ & G/Z_G \arrow[ur] & \end{tikzcd} \quad \forall V \in \mathcal{O}(\mathcal{C})$$

On symmetric fusion cats:

Recall (8.1.12): $\mathcal{C}$ symmetric

$$c_{YX} \circ c_{YX} = id_{X\otimes Y}$$

E.g.'s (9.9.1), (9.9.3):

- Rep(G), G finite.

$$c_{XY}(x \otimes y) = y \otimes x \qquad (x \in X,\ y \in Y)$$

- sVec $=$ Vec$_{\mathbb{Z}/2}$.

$$c_{XY}(x \otimes y) = (-1)^{mn}\, y \otimes x \qquad (x \in X_{(-1)^m},\ y \in Y_{(-1)^n})$$

- Rep(G, z), G finite, $z \in Z_G$, $z^2 = 1$.

$$\begin{cases} zx = (-1)^m x \\ zy = (-1)^n y \\ c_{XY}(x \otimes y) = (-1)^{mn}\, y \otimes x \end{cases} \qquad x \in X,\ y \in Y$$

Plan: $\cancel{1}$, 2, 3, 4, 5, $\cancel{6}$, $\cancel{7}$, 8, 9 + sketch of proofs.

On positive symmetric fusion cats:

Def (9.9.5): $V \in \mathcal{C}$, $S_n \to \mathrm{Aut}_{\mathcal{C}}(V^{\otimes n})$

$$P_\epsilon = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon^{|\sigma|} \sigma \in k[S_n], \qquad \epsilon \in \{\pm 1\}$$

$$S^n V = P_+(V^{\otimes n}), \quad \Lambda^n V = P_-(V^{\otimes n})$$

Recall.

$$X \xrightarrow{u} Y \rightsquigarrow \delta(u) = \mathbb{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{u} Y \otimes X^* \cong X^* \otimes Y \xrightarrow{Y=X} \mathbb{1}$$

$$X = Y \rightsquigarrow \mathrm{Tr}(u) = \mathrm{ev}_X \circ \delta(u)$$

$$\dim(X) = \mathrm{Tr}(1_X)$$

if $X \xrightarrow{u} Y \xrightarrow{v} Z$

$\delta(vu) = \delta(v) \otimes \delta(u)$:

$$\begin{array}{ccccc}
\mathbb{1} & \longrightarrow & X \otimes X^* & \xrightarrow{vu} & Z \otimes X^* \\
\mathbb{1} & & & & \uparrow \mathrm{ev}_Y \\
Y \otimes Y^* \otimes X \otimes X^* & & \xrightarrow{v \otimes u} & & Z \cdot Y^* \cdot Y \cdot X^*
\end{array}$$

On positive symmetric fusion cats:

Exer (9.9.9): $\dim V = \alpha \Rightarrow \begin{cases} \dim \Lambda^n V = \binom{\alpha}{n} = \alpha(\alpha-1)\cdots(\alpha-n+1)/n! \\ \dim S^n V = \binom{\alpha+k-1}{n} . \end{cases}$

Say $u_i : X_i \longrightarrow X_{i+1}$, $i \in \mathbb{Z}/n$

$$\otimes_i u_i : X_1 \otimes \cdots \otimes X_n \longrightarrow X_2 \otimes \cdots \otimes X_1 \triangleq \otimes_i X_i$$

$$ev_1 \circ \delta(u_n - u_1) : \mathbb{1} \longrightarrow X_1 \otimes (X_n^* X_n)(X_{n-1}^* \cdots X_2) \cdot X_1^* \xrightarrow{ev_n \cdots ev_2} X \cdot X^* \longrightarrow \mathbb{1}$$

$$\therefore \operatorname{Tr}(u_n - u_1, X_1) = \operatorname{Tr}(\otimes_i u_i, \otimes_i X_i)$$

Say $\sigma \in S_n$, a cycle. $\begin{cases} X_i = X & \forall i \\ u_i = 1_X \end{cases} \Rightarrow \operatorname{Tr}(\sigma) = \dim X.$

σ has p-cycle $\Rightarrow \operatorname{Tr}(\sigma) = (\dim X)^p$

$\square$

On positive symmetric fusion cats:

Exer (9.9.10): If $\binom{\alpha}{n}$, $\binom{\alpha+n-1}{n}$ are alg int for all $n \in \mathbb{Z}_+ \Rightarrow \alpha \in \mathbb{Z}$.

$Q = x^d + a x^{d-1} + \cdots \in \mathbb{Z}[x]$ min ply of α, $(a \leq 0)$ $(*)$

$\sigma\left(\binom{\alpha}{n}\right) = \binom{\sigma\alpha}{n}$ alg int. $\forall \sigma \in Gal$

$\Rightarrow N\left(\binom{\alpha}{n}\right) = \prod_\sigma \sigma\binom{\alpha}{n} = N(\alpha)\, N(\alpha-1) \cdots N(\alpha-n+1) / (n!)^d \in \mathbb{Z}$

$\prod_\alpha (\sigma\alpha - \lambda)$

$\Rightarrow Q(0)\, Q(1) \cdots Q(n-1) / (n!)^d \in \mathbb{Z}$, $N(\alpha - \lambda) = (-1)^d Q(\lambda)$, $\forall \lambda \in \mathbb{Q}$

$n \gg 1$, $Q(n-1) \leq n^d \Rightarrow b_n = \left| Q(0) \cdots Q(n-1) / (n!)^d \right|$ is decreas $n \gg 1$

$\Rightarrow b_n = 0$, $\exists n$

$\Rightarrow Q$ has a root in $\mathbb{Z}$

$\Rightarrow Q = x + \alpha$

$\Rightarrow \alpha \in \mathbb{Z}$. $\square$

On positive symmetric fusion cats:

Cor (9.9.11): $\mathcal{C}$ symm fusion is integral and $\dim X = \pm \mathrm{FP}(X)$ for all X.

$\dim X \in \mathbb{Z}, \forall X \overset{9.6.2}{\Longrightarrow}$!

Def/Cor (9.9.14): $\mathcal{C}$ symm fusion.

- $\mathcal{C}$ positive if $\dim X = \mathrm{FP}(X)$ for all X.
- Exists unique $\mathbb{Z}/2$ grading $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_{-1}$ with $\underline{\mathcal{C}}_1$ positive
- $X \in \mathcal{C}_{(-1)^m}$, $Y \in \mathcal{C}_{(-1)^n}$ $c^{mod}_{\underline{XY}} := \underline{(-1)^{mn} c_{XY}}$.

Exer (9.9.15): $\dim^{mod} X = (-1)^m \dim X$ if $X \in \mathcal{C}_{(-1)^m}$.

$\underline{\text{Rem.}}$: $\cdot$ $\mathcal{C}$ int $\Rightarrow$ $\mathcal{C}$ w.int $\overset{9.6.5}{\Longrightarrow}$ $\mathcal{C}$ ps unit $\overset{9.6.6}{\Longrightarrow}$ $\exists!\ a_X$ s.t. $\dim_a X = \mathrm{FP}(X)$

$\cdot$ Using c^{mod} vs c $\Longrightarrow$ $\mathcal{C}^{mod}$ $\not\Rightarrow$ pos.

$\cdot$ $(c^{mod})^{mod} = c$.

Plan: ~~1~~, ~~2~~, 3, 4, 5, ~~6~~, ~~7~~, 8, 9 + sketch of proofs.

On braided functors:

Recall (8.1.7): Braided functor: $(F, J) : (\mathcal{C}^1, c^1) \to (\mathcal{C}^2, c^2)$ s.t.

$$\begin{array}{ccc} FX \cdot FY & \xrightarrow{c^2} & FY \cdot FX \\ J \downarrow & & \downarrow J_{YX} \\ F\,XY & \xrightarrow{F(c^1)} & F\,YX \end{array} \qquad F(c^1_{XY})J_{XY} = J_{YX}c^2_{FXFY}$$

Def (9.9.16): $\mathcal{C}$ symm fusion. A symm fiber func is a braided tensor $F : \mathcal{C} \to \mathrm{Vec}$. A super fiber func is a braided tensor $F : \mathcal{C} \to \mathrm{sVec}$. If $\mathcal{C}$ admits a symm fiber func it is Tannakian.

Exer (9.9.18): Tannakian cats are positive. $\dim(X) = \dim(FX) \in \mathbb{Z}_{\geq 0}$

Def/Exer (9.9.19), (9.9.20): Let $F : \mathcal{C} \xrightarrow{I} \mathrm{Vec}$ symm fiber func. Have:

- $A_F = I(\cancel{1}k)$, with I adj of F. $I(k) = \underline{H}(k, k)$ (7.9.10)
  $H_{\mathcal{C}}(X, I(k)) \simeq H_V(F X \otimes k, k)$
  $\simeq H_{\mathcal{C}}(X, \underline{H}(k, k))$
- $\underline{G_F} := \mathrm{Aut}_{\otimes}(F)$.
- $\underline{A_F}$ is a comm algebra in $\mathcal{C}$ and $\underline{G_F \cong \mathrm{Aut}(A_F)}$.
  (8.8.8)

Plan: ~~1~~, ~~2~~, ~~3~~, ~~4~~, ~~5~~, ~~6~~, ~~7~~, ~~8~~, ~~9~~ + sketch of proofs.

Sketches, pt. 1

Thm (9.9.22): All symm fiber funcs are iso.

Claim $\mathrm{Vec} \simeq \mathrm{Mod}_{\mathcal{C}}(A)$, $A = I(k)$, I adj of s.f.f. F,

$\mathcal{G} : \mathrm{Vec} \to \mathrm{Mod}(A) \;:\; V \mapsto \underline{H}(k, V) \in \mathrm{Mod}.$

$\mathcal{U} : \mathrm{Mod}(A) \to \mathrm{Vec} \;:\; M \mapsto H_{\mathcal{C}}(\mathbb{1}, M) \in \mathrm{Vec}$

$$\Rightarrow \mathcal{U}\mathcal{G}(V) = H_{\mathcal{C}}(\mathbb{1}, \underline{H}(k,V)) \simeq H_V(\overset{F\mathbb{1}\otimes k}{\mathbb{1} \otimes k}, V)$$
$$\simeq H_V(k, V) \simeq V.$$

Given $\underline{F} : \mathcal{C} \to \mathrm{Vec}$,

Let $\mathcal{H} : \mathcal{C} \to \mathrm{Vec}$ $\qquad \mathcal{H} : X \mapsto \underline{H_{\mathcal{C}}(\mathbb{1}, X \otimes A)}$

$$X \mapsto H_{\mathcal{C}}(\mathbb{1}, \underline{H}(k, \underline{F}X)) \simeq H_{\mathcal{C}}(\mathbb{1}, \underline{H}(k, \underline{X} \otimes k))$$

(9.9.20)

$$\overset{(7.25)}{\simeq} H_{\mathcal{C}}(\mathbb{1}, X \otimes \underbrace{\underline{H}(k,k)}_{= A.}))$$

Given: $F_i : \mathcal{C} \to \mathrm{Vec}$, $A_i = I_i(k)$ $\quad i \in \{1,2\}$

$\longrightarrow (\dots) \quad A_1 \simeq A_2$

$\Rightarrow F_1 \simeq F_2$

$\square$

Sketches, pt. 2

Thm (9.9.22): $\tilde{F} : \mathcal{C} \cong \operatorname{Rep}(G_F)$.

$H_A(X \otimes A, M) \simeq \mathcal{H}_{\mathcal{C}}(X, M)$, $A = I(k)$ $\forall M$.

- $A \otimes A$ $\otimes$-gen $\mathrm{Bim}_{\mathcal{C}}(A)$: $H_{(A,A)}(A \otimes A, M) \overset{7.8.12}{\simeq} \mathcal{H}_{\mathcal{C}}(\mathbb{1}, M)$ $\neq 0$

- $\mathrm{End}_A(A \otimes A) \simeq \mathcal{H}_{\mathcal{C}}(\mathbb{1}, A \otimes A) \simeq F(A)$ comm. alg. (comm.)

$\therefore$ $M \in \mathrm{Sim}\,\mathrm{Bim}_{\mathcal{C}}(A) \Rightarrow \begin{cases} \mathcal{H}_{\mathcal{C}}(\mathbb{1}, M) \simeq H_{(A,A)}(A \otimes A, M) \quad \text{1 dim'l} \\ M \simeq A \text{ with right action} \\ \text{twisted by } G_F. \end{cases}$

$\Rightarrow \mathrm{SBim}(A) \simeq \mathcal{C}_{G_F}$ $\underline{(2.3.6)}$

- $\mathrm{Vec} \simeq \mathrm{Mod}_{\mathcal{C}}(A) \Rightarrow \mathcal{C}^*_{\mathrm{Vec}} \overset{7.12.2}{=} \mathrm{Fun}(\mathrm{Vec}, \mathrm{Vec}) \overset{7.11.1}{\simeq} \mathrm{Bim}_{\mathcal{C}}(A)$

- $\operatorname{Rep}(G_F) \cong \mathrm{Fun}(\mathrm{Vec}, \mathrm{Vec})^{\mathcal{C}_{G_F}}$

$\Rightarrow \operatorname{Rep}(G_F) \simeq (\mathcal{C}^*_{\mathrm{Vec}})^*_{\mathrm{Vec}} \underset{7.12.11}{\simeq} \mathcal{C}$ $\square$

Sketches, pt. 3

Cor (9.9.25): $\mathcal{C}$ braided equiv to some $\mathrm{Rep}(G, z)$.

$\mathcal{C}$ has $\mathbb{Z}/2$ grading, $\mathcal{C}^{\mathrm{mod}}$ pos.

$\Rightarrow \quad \mathcal{C}^{\mathrm{mod}} \simeq \mathrm{Rep}(G) \qquad \exists\, G$

$\Rightarrow \quad \mathcal{C} \simeq (\mathcal{C}^{\mathrm{mod}})^{\mathrm{mod}} \simeq \mathrm{Rep}(G)^{\mathrm{mod}}$

$\simeq \mathrm{Rep}(G, z)$.

Sketches, pt. 4

Exer(9.9.27): Thm 9.9.22 implies

- $\exists F : \mathcal{C} \to$ sVec super fiber functor
- all super fiber functors are isomorphic
- given $F : \mathcal{C} \to$ sVec get equiv $\tilde{F} : \mathcal{C} \to \mathrm{Rep}(G_F, z_F)$