Tensor Categories, Sections 9.6–9.8

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Section 9.6: Integral and weakly integral fusion cats

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$R = \sum_i \mathtt{FPdim}(b_i) b_i$ reg. elem (fusion), $\mathtt{FPdim}(A) = \mathtt{FPdim}(R)$.

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A is int. $\mathtt{FPdim}(b) \in \mathbb{Z}$ for all $b \in B$

Section 9.6, still

$\mathcal{C}$ fusion $\Rightarrow$ $Gr(\mathcal{C})$ fusion ring

9.6.1. Def: $\mathcal{C}$ fusion is $\begin{cases} \text{w. int.} & \text{if } \mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z} \\ \text{int.} & \text{if } \mathrm{FPdim}(X) \in \mathbb{Z}\ \forall X. \end{cases}$

9.4.1.
$d_X^2 \leq FP(X)^2\ \forall X$
$\dim \mathcal{C} \leq FP(\mathcal{C})$.
$\Sigma\, d_X^2 \leq \Sigma\, FP(X)^2$

9.6.2. Exer: $\mathcal{C}$ spherical fusion, $\dim(X) \in \mathbb{Z}\ \forall X$.
Then $\mathcal{C}$ is int. and $\dim(X) \in \{\pm \mathrm{FPdim}(X)\}\ \forall X$ simple.

Pf: $(\mathcal{C}, \psi)$: $d_X = Tr^L(\psi_X) \in \mathbb{Z}\ \forall$ obj X, $\psi_X : X \xrightarrow{\sim} X^{**}$

$\dim(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} |X|^2 = \sum_X d_X d_{X^*} = \sum_X d_X^2 \in \mathbb{Z}$.

Claim: $\dim(\mathcal{C}) = FP(\mathcal{C})$: (8.20.13) $\Rightarrow$ $Z(\mathcal{C})$ is modular and

$\frac{\dim Z(\mathcal{C})}{FP(Z(\mathcal{C}))} = \frac{(\dim \mathcal{C})^2}{FP(\mathcal{C})^2}$ (9.3.4) (7.16.6) $\Rightarrow$ can assume $\mathcal{C}$ modular.

$\mathcal{C}$ mod
$\Rightarrow \exists X$ s.t. $FP(Z) = \frac{1}{d_X} s_{ZX}\ \forall Z \in \mathcal{O}(\mathcal{C})$.

$\Rightarrow FP(\mathcal{C}) = \sum_{\mathcal{O}(\mathcal{C}) \ni Z} FP(Z)^2 = \sum_Z \frac{s_{ZX}}{d_X} \frac{s_{Z^*X}}{d_X} \overset{8.14.2}{=} \frac{\dim(\mathcal{C})}{d_X^2} \Rightarrow R_\mathcal{C} = \frac{\dim \mathcal{C}}{FP(\mathcal{C})} \in \mathbb{Z}$

& $R_\mathcal{C} \leq 1$ (9.4.1) $\Rightarrow \dim \mathcal{C} = FP(\mathcal{C}) \Rightarrow d_X^2 = FP(X)^2\ \forall X$. $\square$

9.6.3. Exer: The cat $\mathcal{C}_2(q)$, q primitive 8^{th} root of unity w. int. but not int. The categories $\mathcal{C}_k(q)$ are not w. int. for $k > 2$.

Section 9.6, still

$FP(\mathcal{C}) = \dim(\mathcal{C})$

9.6.5. Prop: $\mathcal{C}$ w. int. fusion over $\mathbb{C}$. Then $\mathcal{C}$ pseudo unitary.

(9.4.2) $\frac{\dim \mathcal{C}}{FP\mathcal{C}} \leq 1$ alg. int. $D := \dim(\mathcal{C})$, $D_1 = D, \dots, D_N = \dim(g_N(D))$

$g_1, \dots, g_N \in Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \Rightarrow \frac{\dim(g_i(\mathcal{C}))}{FP(g_i(\mathcal{C}))} \leq 1 \Rightarrow \prod_i \frac{D_i}{FP(g_i(\mathcal{C}))} \leq 1 \Rightarrow \frac{D}{FP(\mathcal{C})} = 1$.

9.6.6. Cor: $\mathcal{C}$ w. int. fusion over $\mathbb{C}$.
Then $\exists! a_X : X \cong X^{**}$ s.t. $d_X = \text{FPdim}(X)$ for all X. (9.5.1)

9.6.7. Cor: H semisimple Hopf alg over k, char$(k) = 0$. Then $S^2 = id$.

$Rep(H)$ is int $(FP(X) = \dim_k(X))$ → fin. order, $\forall V$ simple

$\Rightarrow \exists!$ sph str: given by gp-like u with $uxu^{-1} = S^2(x)$ & $Tr_V(u) = \dim V$

$\Rightarrow \sum_{i=1}^{m} \lambda_i = m \Rightarrow \lambda_i = 1 \ \forall i \Rightarrow u = id.$ □

9.6.9. Prop: $\mathcal{C}$ w. int. fusion. Then

- $\forall X, \exists n_X \in \mathbb{Z}$ with $\text{FPdim}(X) = \sqrt{n_X}$ (3.5.7)
- $\deg : \mathcal{O}(X) \to \mathbb{Q}_+^\times / (\mathbb{Q}_+^\times)^2$ with $\deg(X) = [\text{FPdim}(X)^2]$ is a grading.

9.6.10. Cor: $\mathcal{C}$ fusion with $\text{FPdim}(\mathcal{C})$ odd integer. Then $\mathcal{C}$ is int. (3.5.8)

Section 9.6, still

9.6.11. Prop: $\mathcal{C}$ int. fusion and $\mathcal{M}$ indecomposable $\mathcal{C}$-mod cat. Then $\mathcal{C}^*_{\mathcal{M}}$ int. fusion.

$F = \text{Forg} : Z(\mathcal{C}) \longrightarrow \mathcal{C}$ s.t. •) preserves FP.

(7.13.11) ••) surj (any $X \in \mathcal{O}(\mathcal{C})$ is subobj. of $F(Y)$).

(7.16.2) $Z(\mathcal{C}^*_{\mathcal{M}}) \cong Z(\mathcal{C}) \Rightarrow Z(\mathcal{C}^*_{\mathcal{M}})$ is int

$\Rightarrow \forall X \in \mathcal{O}(\mathcal{C}^*_{\mathcal{M}})$, $\exists Y, X'$ s.t. $Y = X + X'$

$\text{FP}(Y) \in \mathbb{Z} \overset{(3.5.6)}{\Longrightarrow} \text{FP}(X), \text{FP}(X') \in \mathbb{Z}$. $\square$

9.6.12. Exer: $\mathcal{C}$ w. int. fusion, then $\mathcal{C}_{\text{ad}}$ is int. fusion.

9.6.13. Exer: $\mathcal{C}$ fusion, $\mathcal{D}$ full abelian subcat st $X \in \mathcal{D}$ iff $\text{FPdim}(X) \in \mathbb{Z}$. Then $\mathcal{D}$ is fusion.

Section 9.7: Group-theoretical fusion cats

Recall (5.11.1): $\mathcal{C}$ ptd fusion if every simple object is invertible.
E.g. (2.3.8): $\mathcal{C} = \mathrm{Vec}_G^\omega$, G finite gp, $\omega \in Z^3(G, k^\times)$ is ptd fusion.

$\mathrm{Vec}_G^\omega \ni V = \bigoplus_g V_g$ G-graded, G fin. gp. 1·1·1·1·1

assoc. a^ω comes from $\omega \in Z^3(G, k^\times)$: $d\omega = 0 \rightsquigarrow$ pentagon

simple objects $\{\delta_g, g\in G\}$ with $(\delta_g)_x = \begin{cases} k & x=g \\ 0 & \text{o.w.} \end{cases}$

$\omega = 0$: assoc. are identities!

9.7.1. Def: $\mathcal{C}$ fusion is gp theor if $\mathcal{C}^*_{\mathcal{M}} \cong \mathrm{Vec}_G^\omega$ for some indec $\mathcal{C}$-mod $\mathcal{M}$.
9.7.2. E.g.: Classification of Vec_G^ω-module cats:

(7.4.10) $\mathcal{C} = \mathrm{Vec}_G^0$. $\mathcal{M}$ indec. $\mathcal{C}$-mod cat $\Longleftrightarrow (\mathcal{M}, F, \eta)$

$F_g : \mathcal{M} \to \mathcal{M}$, $F_g(M) = \delta_g \otimes M$, $\eta_{gh} : F_g \circ F_h \simeq F_{gh}$

s.t. $\eta_{gh,k} \circ \eta_{gh} = \eta_{g,hk}\eta_{hk}$. $(\eta \in Z^2(G, A))$

Section 9.7, still

Given $\mathcal{M}$ $\rightsquigarrow$ $\exists\, L < G$ s.t. $\mathcal{O}(\mathcal{M}) = G/L$, L unique up to conj.

assoc. of m $\rightsquigarrow$ $\psi : G \times G \longrightarrow \mathrm{Fun}(G/L, k^\times) =: \mathrm{coind}_L^G k^\times$

$$\underline{\psi}(x,y)(b) = m_{x,y,y^{-1}x^{-1}b}$$

(pent.) $\psi \in Z^2(G, \mathrm{coind}_L^G k^\times)$

but $H^2(G, \mathrm{coind}_L^G k^\times) \simeq H^2(L, k^\times)$

$\therefore$ (Vec_G) - mod indu. $\mathcal{M} = \mathcal{M}(L,\psi)$, $L < G$, $\psi \in H^2(L, k^\times)$

Similarly $\omega \neq 0$: $d\psi = \omega|_{L\times L\times L}$ (L,ψ), $\psi \in C^2(L, k^*)$

Section 9.7, still

9.7.3. Rem: $\mathcal{M}(L,\psi) \sim_{Mor} \mathcal{M}(L',\psi')$ iff

$$L' = {}^{g}L = gLg^{-1} \quad \text{and} \quad \psi' = \psi^{g} = \psi({}^{g}\bullet, {}^{g}\bullet).$$

9.7.4 – 9.7.6: skip
9.7.7. Rem: gp theor fusion cats are int.

9.7.8. Def: $\mathcal{D}$ is a *quotient* of a fusion $\mathcal{C}$ if $\exists F : \mathcal{C} \to \mathcal{D}$ surjective.

Recall 4.3: If $\mathbb{1} = \oplus_i \mathbb{1}_i$ then $\mathcal{C}_{ij} = \mathbb{1}_i \otimes \mathcal{C} \otimes \mathbb{1}_j$ are the components of $\mathcal{C}$.
9.7.9. Prop:

(i) Subcats of gp theor fusion is gp theor fusion.

(ii) Components in a quotient of a gp theor fusion is gp theor fusion.

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Recall: Section 3.6, 4.14

Let (A, B) a unital based ring

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A is nilp if $A = A^{(0)} \supset A^{(1)} \supset \cdots \supset A^{(n)} = \mathbb{Z}.1$ for some n,
with $A^{(1)} = A_{ad}$ and $A^{(k)} = (A^{(k-1)})_{ad}$

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$\mathcal{C}$ is nilp if $\mathcal{C} = \mathcal{C}^{(0)} \supset \mathcal{C}^{(1)} \supset \cdots \supset \mathcal{C}^{(n)} = \text{Vec}$ for some n,
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Fact: $\mathcal{C}$ is nilp iff exists $\mathcal{C}_0 = \mathrm{Vec} \subsetneq \mathcal{C}_1 \subsetneq \cdots \subsetneq \mathcal{C}_n = \mathcal{C}$ with $\mathcal{C}_i$ faithful G_i-grd with trivial comp $\mathcal{C}_{i-1}$. $\mathcal{C}$ cycl nilp cat if G_i cyclic.

Section 9.8, still

9.8.1. Def: $\mathcal{C}$ fusion is $\begin{cases} \text{w gp theor} & \text{if } \sim_{Mor} \text{ to a nilp cat} \\ \text{solv} & \text{if } \sim_{Mor} \text{ to a cycl nilp cat} \end{cases}$

9.8.2. Rem: $\text{FPdim}(\mathcal{A}) \in \mathbb{Z}$ for all w gp theor fusion cats.

9.8.3. Lem: G fin, $\mathcal{A}$ a G-ext of $\mathcal{A}_0 \sim_{Mor} \mathcal{B}_0$.
Then exists a G-ext $\mathcal{B}$ of $\mathcal{B}_0$ with $\mathcal{B} \sim_{Mor} \mathcal{A}$.

$$\mathcal{A} = \mathcal{A}_0 \oplus \cdots$$
$$\wr_{Mor} \qquad \wr_{Mor}$$
$$\Rightarrow \exists\, \mathcal{B} = \mathcal{B}_0 \oplus \cdots$$

$A \in \mathcal{A}_0 \leq \mathcal{A}$

$\mathcal{B}_0 \cong \text{Bimod}_{\mathcal{A}_0}(A)$

$\Rightarrow \mathcal{B} \cong \text{Bimod}_{\mathcal{A}}(A).$

Section 9.8, still

9.8.4. Prop: The class of w gp theor fusion cats is closed under:

- G-extensions,
- G-equivariantizations,
- categorical Morita,
- tensor prods,
- centers,
- subcats,
- components of quotient cats.

$$\mathcal{D} = \mathcal{C}_0' \oplus \bigoplus_{g \neq 1} \mathcal{D}_g$$

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9.8.5. Prop: (i) The class of solv fusion cats is closed under:

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9.8.6. Question: Is $\mathrm{Rep}(H)$ w gp theor if H is ss fin dim Hopf algebra?