Chapter 9: Fusion cats

9.1: We will use the absence of deformations to prove various "finiteness" theorems.

First some background: Given an algebra A, we define the Hochschild complex

$$\cdots \to \mathrm{Hom}_k(A^{\otimes(n-1)}, A) \xrightarrow{d} \mathrm{Hom}_k(A^{\otimes n}, A) \xrightarrow{d} \mathrm{Hom}_k(A^{\otimes(n+1)}, A) \to \cdots$$

with
$$df(x_1, \dots, x_{n+1}) = x_1 f(x_2, \dots, x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) + (-1)^{n+1} f(x_1, \dots, x_n) x_{n+1}$$

$\rightsquigarrow d \circ d = 0$ (calculation)

Thus, $HH^1(A)$ Hochschild cohomology

Properties: • $HH^1(A) = \mathcal{D}er(A)$

• A Frobenius algebra, $m : A\otimes A \to A$,

$\Delta : A \to A\otimes A \in Hom_{(A,A)\text{-mod}}$

$\varepsilon : k \to A \quad \in Hom_{(A,A)\text{-mod}}$

A is separable if $m\circ\Delta : A \to A$ is isomorphism

$\Updownarrow$

$v = m\circ\Delta(1)$ is invertible

$\Rightarrow x \mapsto \Delta(v^{-1}x)$ is inverse.

$(HH^0(A) = k)$

In this case, we have $HH^n(A) = 0$ for $n > 0$

Pf:

1) $HH^1(A) = \dfrac{Ker(d_1)}{Im(d_0)} = 0.$

$f : \quad x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2 = 0$

2) Let $f \in C^n(A)$ s.t. $d_n f = 0$. Define $\tilde{f} \in C^{n-1}$

$\tilde{f}(x_1, \ldots, x_{n-1}) := v^{-1} 1_1 f(1_2, x_1, \ldots, x_{n-1})$

Then: (note u is central!)

$$u\, d_{(n-1)} \tilde{f}(x_1, \dots, x_n)$$

$$= x_1 1_1 f(1_2, x_2, \dots, x_n) - 1_1 f(1_2, x_1 x_2, \dots, x_n)$$

$$+ \dots + (-1)^n 1_1 f(1_2, x_1, \dots, x_{n-1}) x_n$$

$$d_n f = 0 \Rightarrow 1_2 f(x_1, \dots, x_n) - f(1_2 x_1, \dots, x_n)$$

$$+ \sum_{i=1}^{n-1} (-1)^{i+1} f(1_2, \dots, x_i x_{i+1}, \dots, x_n)$$

$$+ (-1)^{n+1} f(1_2, x_1, \dots, x_{n-1}) x_n = 0$$

$$(-1)^{n-1} 1_1 f(1_2, x_1, \dots, x_{n-1} x_n)$$

$$= 1_1 1_2 f(x_1, \dots, x_n) - 1_1 f(1_2 x_1, \dots, x_n)$$

$$+ \dots + (-1)^{n-1} 1_1 f(1_2, x_1, \dots, x_{n-2} x_{n-1}, x_n)$$

$$+ (-1)^{n+1} 1_1 f(1_2, x_1, \dots, x_{n-1}) x_n$$

$$u\, d_{(n-1)} \tilde{f}(x_1, \dots, x_n) = 1_1 1_2 f(x_1, \dots, x_n)$$

$$+ \underbrace{x_1 1_1 f(1_2, x_2, \dots, x_n) - 1_1 f(1_2 x_1, \dots, x_n)}_{= 0 \text{, bc } x_1 1_1 \otimes 1_2 = 1_2 \otimes 1_2 x_1}$$

Davydov-Yetter Cohomology: Categorification of Hochschild

$\mathcal{C}, \mathcal{C}'$ multitensor over k, $F : \mathcal{C} \to \mathcal{C}'$ tensor functor
(cf. Reconstruction theory and fiber functors)

Define $T_n : \mathcal{C}^n \to \mathcal{C}$ by $T_n(X_1, \dots, X_n) := X_1 \otimes \dots \otimes X_n$

$C^n_F(\mathcal{C}) := \mathrm{End}(T_n \circ F^n) \quad (C^0_F(\mathcal{C}) = \mathrm{End}(1_{\mathcal{C}'}))$

Define differential $d : C^n_F(\mathcal{C}) \to C^{n+1}_F(\mathcal{C})$

$$df = \mathrm{id} \otimes f_{2,\dots,n+1} - f_{12,\dots,n+1} + f_{1,23,\dots,n+1}$$
$$\dots + (-1)^{n+1} f_{1,\dots,n} \otimes \mathrm{id}. \quad (\text{using } F(X_1 \otimes X_2) \cong F(X_1) \otimes F(X_2))$$

→ $d^2 = 0$, so $(C^\bullet_F(\mathcal{C}), d)$ is a complex.
→ Davydov-Yetter cohomology if $\mathcal{C} = \mathcal{C}'$, $F = \mathrm{id}$.

Exercise: (i) $\mathcal{C}$ indecomp multitensor cat. $\Rightarrow H^0_F(\mathcal{C}) = k$.
(ii) $H^1_F(\mathcal{C})$: Lie algebra of derivations of F as a tensor functor (cf. Hochschild)

(iii) Show $H^2_F(\mathcal{C})$ parametrises first order deformations of F as a tensor functor

$$\begin{tikzcd}
F(X)\otimes F(Y)\otimes F(Z) \arrow[r, "J_{Y,Z}"] \arrow[d, "J_{X,Y}\otimes \mathrm{id}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X)\otimes F(Y\otimes Z) \arrow[d, "J_{X,Y\otimes Z}"] \\
F(X\otimes Y)\otimes F(Z) \arrow[r, "J_{X\otimes Y,Z}"'] & F(X\otimes Y\otimes Z)
\end{tikzcd}$$

$$\varepsilon f_{1,2}\otimes \mathrm{id} + \varepsilon f_{12,3} = \varepsilon\, \mathrm{id}\otimes f_{2,3} + \varepsilon f_{1,23}$$

$$d_2 f = 0 \iff \mathrm{id}\otimes f_{2,3} - f_{12,3} + f_{1,23} - f_{1,2}\otimes \mathrm{id}$$

(iv) $H^3(\mathcal{C})$ parametrises first order deformations of $\mathcal{C}$ as multitensor category

$$\begin{tikzcd}
 & ((W\otimes X)\otimes Y)\otimes Z \arrow[dl] \arrow[dr] & \\
(W\otimes(X\otimes Y))\otimes Z \arrow[d] & & (W\otimes X)\otimes(Y\otimes Z) \arrow[d] \\
W\otimes((X\otimes Y)\otimes Z) \arrow[rr] & & W\otimes(X\otimes(Y\otimes Z))
\end{tikzcd}$$

$\mathcal{C}$ multifusion, $A := \underline{\mathrm{Hom}}_{\mathcal{C} \boxtimes \mathcal{C}^{op}}(1, 1)$

canonical Frobenius algebra (Seen by Marcelo)

Why is it Frobenius? (7.19, 7.20)
↳ Proposition 7.20.1

Proposition (7.22.7):

$$C^\bullet(A) \cong C^\bullet(\mathcal{C})$$

Pf: Long and Technical.

Theorem: Let k be alg closed, char$(k) = 0$, $\mathcal{C}$ multifusion. Then $H^n(\mathcal{C}) = 0$ for all $n > 0$

Pf: Equivalent to $H^n(A) = 0$, but by Corollary 7.21.19, A is separable. ($m \circ \Delta$ is auto) $\square$
(Proof in book seems redundant)

Theorem: A multifusion cat does not admit nontrivial deformations. In particular, number of mfusion cats w/ given Grothendieck ring is finite.

Prf: $H^3(\mathcal{C})=0$ By previous theorem.

X: all admissible associativity constraints with Grothendieck ring $\mathcal{C}$
affine algebraic variety (6j-symbols) → coordinatise.

acted upon by group of twists G
↪ automorphisms of H_{ij}^k
$\mathrm{Hom}(X_k, X_i \otimes X_j)$

For $x \in X$: $\mathcal{C}_x$ the assoc fusion cat.
$f_x: G \to X: x \mapsto g \cdot x$

$(df_x)_1: \mathrm{Lie}(G) \to T_x X$, $T_x X$ tangent space

$T_x X: Z^3(\mathcal{C}_x)$ (already done)
$(df_x)_1(\mathrm{Lie}(G)): B^3(\mathcal{C}_x)$ (???)

Alg Geometry?

$\Rightarrow (df_x)_1$ is surjective $\Rightarrow Gx$ are open in X
$\Rightarrow$ finitely many orbits. (variety) $\square$

Theorem A tensor functor between multifusion cats does not have nontrivial first order def
-> Number of functors is finite
Pf: Analogous, but wrong. $H^n_F(\mathcal{C})=0$ as well
(weak Hopf algebras, paper by one of authors)
Second part.

functor ⇝ matrix with non-negative integer entries.
and it conserves FPdim (Talk by Gert) □
so only finitely many options for this matrix.

Corollary: (i) A module cat M over a multifusion cat does not admit non trivial def. => ...
(ii)
9.1.6 sucks and I don't understand.

Corollary 9.1.7: Number of iso classes of semisimple Hopf algebras of dimension d over alg closed field k

Pf: finitely many fusion rings of FPdim d
Each of them has fin categorific: 9.1.4
finitely many fiber functors.

<u>Corollary</u>: Any mfusion cat, any tensor f and any semisimple Hopf alg defined over alg number field.

<u>Pf</u>: Jesus. G_X is irred comp of variety over $\mathbb{Q}$ $\Rightarrow$ defined over $\bar{\mathbb{Q}}$, point with coordinates in $\bar{\mathbb{Q}}$ $\square$

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§9.2 Induction to the center.

$\mathcal{C}$ fusion cat, $Z(\mathcal{C})$ $Obj: (X, \gamma)$

$\gamma_Y : X \otimes Y \xrightarrow{\sim} Y \otimes X$

<u>Lemma</u>: Consider $\mathcal{C}$ as $\mathcal{C} \boxtimes \mathcal{C}^{op}$ module

$$\Rightarrow \quad \underline{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{op}}(V, W) \cong \bigoplus_{X \in O(\mathcal{C})} X \boxtimes ({}^*V \otimes {}^*X \otimes W)$$

<u>Pf</u>: messing around with adjunction iso's, and def of $\underline{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{op}}$, since

$$(Y_1 \boxtimes Y_2) \cdot V = Y_1 \otimes V \otimes Y_2 \qquad \square$$

Proposition: Let $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ and $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ adjoint (I is induction)

$$FI(Y) \cong \bigoplus_{X \in \mathcal{O}(\mathcal{C})} X \otimes Y \otimes X^*$$

Proof: $I(Y) = \underline{\mathrm{Hom}}_{\mathcal{Z}(\mathcal{C})}(1, Y)$ (easy to see)

$$\mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, \underline{\mathrm{Hom}}_{\mathcal{Z}(\mathcal{C})}(1, Y))$$
$$\cong \mathrm{Hom}(X \cdot 1, Y)$$

Prop 7.12.28: $\underline{\mathrm{Hom}}_{\mathcal{C} \boxtimes \mathcal{C}^{op}}(Y, 1) \otimes 1 \cong {}^*\underline{\mathrm{Hom}}_{\mathcal{Z}(\mathcal{C})}(1, Y) \otimes 1$

$$\underline{\mathrm{Hom}}_{\mathcal{C} \boxtimes \mathcal{C}^{op}}(Y, 1) \cong \bigoplus_{X \in \mathcal{O}(\mathcal{C})} X \boxtimes ({}^*Y \otimes {}^*X)$$

$$\cong \qquad\qquad \cong$$

$${}^*\underline{\mathrm{Hom}}_{\mathcal{Z}(\mathcal{C})}(1, Y) \cong \bigoplus_{X \in \mathcal{O}(\mathcal{C})} X \boxtimes {}^*Y \otimes {}^*X$$

$$\Downarrow$$

$$F(I(Y)) \cong \bigoplus_{X \in \mathcal{O}(\mathcal{C})} X \otimes Y \otimes X^*$$

or ${}^*X$

don't really know

<u>Remark</u>: $I(1)$ is algebra

$$FI(1) \cong \bigoplus_{X \in \mathcal{O}(\mathcal{C})} \underbrace{X \otimes X^*}_{\to \text{ algebras}}$$

$m = id_X \otimes ev_X \otimes id_{X^*}$

<u>Lemma</u>: central structure on $I(1)$ is

$$\rho_Y : Y \otimes \left(\bigoplus_{X \in \mathcal{O}(\mathcal{C})} {}^*X \boxtimes X \right) \xrightarrow{\sim} \left(\bigoplus_{X \in \mathcal{O}(\mathcal{C})} {}^*X \boxtimes X \right) \otimes Y$$

Pf: Diagram 7.39. $\square$

<u>Exercise</u>: Prove that invertible subobjects of $I(1)$ form a group isomorphic to $\mathrm{Aut}_\otimes(id_\mathcal{C}) = \{\text{nat iso's } \rho_X : X \xrightarrow{\sim} X\}$

Given (Z, γ) invertible subobject of 1, we get:

Corollary 8.22.8, 8.22.9

$$Y \xrightarrow{\sim} Y \otimes X \otimes X^* \to X \otimes Y \otimes X^*$$
$$\to Y \otimes X \otimes X^*$$
$$\to Y$$

Let $\mathcal{C}$ be fusion, The $Z(\mathcal{C})$ is nondegenerate

8.20.14

Why subobjects of $I(1)$?

§9.3: Duality for fusion categories.

$\mathcal{C}$ module cat $\mathcal{M}$ is exact iff semisimple

$\Rightarrow \mathcal{C}^*_{\mathcal{M}}$ also fusion?

$\hookrightarrow \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$

See section 7.21

<u>Lemma</u> For any natural iso. $\phi_X : X \xrightarrow{\sim} X^{**}$

$$\Phi := \bigoplus_{X \in \mathcal{O}(\mathcal{C})} ({}^*\phi_X)^{-1} \otimes \phi_X : \bigoplus_{X \in \mathcal{O}(\mathcal{C})} {}^*X \otimes X \xrightarrow{\sim} \bigoplus_{X \in \mathcal{O}(\mathcal{C})} X^* \otimes X^{**}$$

is an iso between $I(1)$ and $I(1)^{**}$

<u>Pf</u>: Commutes with the central structure

$$\Phi^{\boxtimes} : \bigoplus_{X \in \mathcal{O}(\mathcal{C})} {}^*X \boxtimes X \xrightarrow{\sim} \bigoplus_{X \in \mathcal{O}(\mathcal{C})} X^* \boxtimes X^{**}$$

$$\begin{tikzcd}
Y \otimes I(1) \arrow[r, "\rho_Y"] \arrow[d, "\Phi^{\boxtimes}"'] & I(1) \otimes Y \arrow[d, "\Phi^{\boxtimes}"] \\
Y \otimes I(1)^{**} \arrow[r, "\rho_Y^{**}"'] & I(1)^{**} \otimes Y
\end{tikzcd}$$

Theorem: Center of fusion cat is a fusion cat.

Pf: $\mathcal{C}$ is fusion cat. $\mathcal{Z}(\mathcal{C})$ is finite tensor cat
(End of finite tensor cat) (to show: 1 is proj.)
is fin multitensor. $\Leftrightarrow$ $\mathcal{C}$ is semisimple

Claim $I(\underline{1})$ is projective

$Hom(-, I(1)) \cong Hom(F(-), 1)$

$F: \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is exact, $\mathcal{C}$ is semisimple, so
$\underline{1}$ is injective, so $Hom(-, I(1))$ is exact.
$\Rightarrow I(1)$ injective $\Rightarrow I(1)$ projective (6.1.3)

$$Tr^L(\underline{\Phi}) = \bigoplus_{X \in \mathcal{O}_{\mathcal{C}}} Tr^L(({}^*\phi_X)^{-1})\, Tr^L(\phi_X) = \bigoplus_{X \in \mathcal{O}_{\mathcal{C}}} |X|^2 = \dim(\mathcal{C})$$

(Section 7.21)
coev is mono
$\Rightarrow Tr^L(\underline{\Phi}) \neq 0 \Rightarrow \underline{1}$ "$\in$" $I(1) \otimes I(1)^*$
$\underline{1}$ is projective as subobject. $\Rightarrow \mathcal{Z}(\mathcal{C})$ is semisimple.

$$Tr^L(\underline{\Phi}) : \quad \underline{1} \xrightarrow{coev} I(1) \otimes I(1)^* \to \dots$$

iso   mono

Corollary $\mathcal{C}$ fusion category, $\mathcal{M}$ semisimple $\mathcal{C}$-module category. Then $\mathcal{C}_{\mathcal{M}}^*$ is multifusion

Pf: again semisimplicity, but $Z(\mathcal{C}_{\mathcal{M}}^*)$ is fusion
$F: Z(\mathcal{C}_{\mathcal{M}}^*) \to \mathcal{C}_{\mathcal{M}}^*$ is surj $\Rightarrow 1$ is projective ($F(1_{Z_\mathcal{C}})$)

Proposition: In any fusion cat $\mathcal{C}$ we have

$$\dim(Z(\mathcal{C})) = \dim(\mathcal{C})^2$$

Proof: Pivotal structure (section 7.21) $\square$

Corollary: $\mathcal{C}$ ribbon fusion $\Rightarrow \frac{\dim(\mathcal{C})}{\dim(X)}$ alg integer for X simple

Pf: $\mathcal{C} \subseteq Z(\mathcal{C})$ $\rightsquigarrow$ modular, so $\frac{\dim(\mathcal{C})^2}{\dim(X)^2}$ alg int.

$\Rightarrow \frac{\dim(\mathcal{C})}{\dim(X)}$ alg integer $\square$

Corollary: G finite, V irrep. $\Rightarrow \dim_k(V) \mid |G|$

Theorem: $\mathcal{C}$ spherical fusion cat

$$\dim(\mathcal{C}) = 1 + \sum_{\substack{Z \in \mathcal{O}(Z(\mathcal{C})) \\ Z \neq 1}} [F(Z):1] \dim(Z)$$

Pf: $\dim(\mathcal{C}) = \dim(\sum X \otimes X^*) = \dim(\sum_{Z \in I} Z) = \dim(1 + \sum [F(Z):1] \dim Z)$
$= 1 + \sum [F(Z):1] \dim Z$

Proposition: $\mathcal{C}$ fusion cat, $\mathcal{M}$ indecomposable $\mathcal{C}$-module cat.

$$\dim(\mathcal{C}) = \dim(\mathcal{C}^*_{\mathcal{M}})$$

Pf: $Z(\mathcal{C}) \cong Z(\mathcal{C}^*_{\mathcal{M}})$

$\mathcal{C}^*_{\mathcal{M}}$ is ~~mult~~ fusion

$\hookrightarrow \mathcal{M}$ indecomposable.

$\Rightarrow \dim(\mathcal{C})^2 = \dim(\mathcal{C}^*_{\mathcal{M}})^2$

and Proposition 7.21.14 $|X|^2$ totally positive.

Lemma: $Gr(Z(\mathcal{C})) \otimes \mathbb{Q} \to Z(Gr(\mathcal{C})) \otimes \mathbb{Q}$ is surjective.

Pf: $I: \mathcal{C} \to Z(\mathcal{C}) \rightsquigarrow I: Gr(\mathcal{C}) \to G(Z(\mathcal{C}))$

$$F(I(X)) = \sum_{Y \in O(\mathcal{C})} Y X Y^* \Rightarrow X \sum_{X \in O(\mathcal{C})} Y Y^*$$

$\hookrightarrow$ self adjoint (w/ resp. to -din of the spaces) positive definite operator!

$\Rightarrow$ invertible. $\square$

Theorem: $\mathcal{C}$ fusion cat, L irred rep of $Gr(\mathcal{C})$

$\Rightarrow \exists$ root of unity ξ s.t. for any obj $X \in Gr(\mathcal{C})$
$Tr(X, L) \in \mathbb{Z}[\xi]$

Pf: assume wlog spherical $\Rightarrow \mathcal{Z}(\mathcal{C})$ is modular

$\rightarrow e_L = \sum_{Y \in \mathcal{O}(\mathcal{C})} Tr(Y, L) Y^*$ is proportional to
a primitive central idempotent. $\square$

Proof: $A \otimes_{\mathbb{Z}} \mathbb{C}$ is semisimple.
And central like 3.1.8 ($\sum b_i x b_i^*$ is central)
$A = \bigoplus_i A_i \rightarrow$ ideals (simple)

Corollary: Any irrep of $Gr(\mathcal{C})$ is defined over some cyclotomic field. In particular, for any homomorphism $\phi: Gr(\mathcal{C}) \rightarrow \mathbb{C}$ and any object $x \in \mathcal{C}$ we have $\phi(x) \in \mathbb{Q}[\xi]$

Pf: Brauer group of $\mathbb{Q}^{ab}$ is trivial
$\rightarrow$ if split, split over finite field extension $\square$

Corollary: $\mathcal{C}$ fusion cat.

$\exists\ \zeta$: $\quad FPdim(X) \in \mathbb{Z}[\zeta]$

9.4 Pseudo-unitary fusion cats.

Prop: $\mathcal{C}$ fusion cat over $\mathbb{C}$

$$|X|^2 \leq FPdim(X)^2$$

$\Rightarrow dim(\mathcal{C}) \leq FPdim(\mathcal{C})$

Pf: Spherical. $\rightarrow |X|^2$ is eigenvalue of $N_X N_{X^*}$

$\Rightarrow |X|^2 \leq FPdim(X \otimes X^*) = FPdim(X)^2$

Prop: $\dfrac{dim(\mathcal{C})}{FPdim(\mathcal{C})}$ is algebraic integer. ≤ 1

Pf: Again spherical. and modular by $Z(\mathcal{C})$ $\rightarrow Tr(c_{Y,X} c_{X,Y})$

$h_X : Y \mapsto \dfrac{s_{XY}}{dim(X)}$ $\quad Y \in \mathcal{O}(\mathcal{C})$ $\qquad S = (s_{xy})$

$\Rightarrow \exists X \in \mathcal{O}(\mathcal{C})$ s.t

$$\mathrm{FPdim}(Z) = \frac{S_{XZ}}{\dim(X)}$$

$$\mathrm{FPdim}(\mathcal{C}) = \sum_Z \mathrm{FPdim}(Z)^2 = \sum_Z \frac{S_{ZX} S_{Z^*X}}{d_X d_X} = \frac{\dim(\mathcal{C})}{\dim(X)^2}$$

$$\Rightarrow \frac{\dim(\mathcal{C})}{\mathrm{FPdim}(\mathcal{C})} = d_X^2 \to \text{algebraic integer.}$$

<u>Definition</u>: $\mathcal{C}$ over $\mathbb{C}$ is pseudo-unitary if $\dim(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C})$

$\to |X|^2 = \mathrm{FPdim}(X)^2$

<u>Examples</u>: $\mathrm{Rep}(G)$, G finite

<u>Exercises</u>: <u>no</u>.

<u>Remark</u> Hermitian category $\rightsquigarrow$ unitary categories.

9.5 Canonical Spherical Structure.

$\mathcal{C}$ fusion over $\mathbb{C}$

$g_X : X \xrightarrow{\sim} X^{****}$ tensor iso

$a_X : X \xrightarrow{\sim} X^{**}$ iso s.t

$a_{X^{**}} \circ a_X = g_X$

$b^V_{XY} : \mathrm{Hom}_{\mathcal{C}}(V, X \otimes Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(V^{**}, X^{**} \otimes Y^{**})$

$$a_X \otimes a_Y = \bigoplus_{V \in O(\mathcal{C})} b^V_{XY} \otimes a_V$$

a_X is tensor iff $b^V_{XY} = \mathrm{id}$, $(b^V_{XY})^2 = \mathrm{id}$.

$N^V_{XY} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(V, X \otimes Y)$, $T^V_{XY} = \mathrm{Tr}(b^V_{XY})$

$|T^V_{XY}| \leq N^V_{XY}$ (because $(b^V_{XY})^2 = \mathrm{id}$.)

$T^V_{XY} = N^V_{XY}$ iff $b^V_{XY} = \mathrm{id}$.

$d_X = \mathrm{Tr}(a_X)$, $|X|^2 = |d_X|^2$

<u>Proposition</u> pseudo-unitary fusion cat admits unique sph structure s.t. $d_X = \mathrm{FPdim}(X)$.

Pf:

$$\begin{tikzcd} V \hookrightarrow X\otimes Y \arrow[r, "\sigma_{X\otimes Y}"] \arrow[d, "\parallel"'] & X\otimes Y \, \frac{|\cdot d_X d_Y|^2}{(d_X d_Y)^2} \arrow[d, "\cup"] \\ V \hookrightarrow X\otimes Y \arrow[r, "\sigma_X\otimes\sigma_Y"'] & X\otimes Y \, \frac{|d_X|^2 |d_Y|^2}{d_X^2 d_Y^2} \end{tikzcd}$$

$a \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$

$$\begin{tikzcd} F(V) \arrow[r, "a"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & F(V) \arrow[d, "F(f)"] \\ F(W) \arrow[r, "a"'] & F(W) \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "\sigma_V"] \arrow[d] & V \, \frac{|d_V|^2}{d_V^2} \arrow[d] \\ X\otimes Y \arrow[r, "\sigma_{X\otimes Y}"'] & X\otimes Y \, \frac{|d_X d_Y|^2}{(d_X d_Y)^2} \end{tikzcd}$$

$1\,!$

$$f(\tilde{e}^2) = f'(\tilde{e})$$

$$\tilde{e}^2 = v$$

$$\Rightarrow (\tilde{e} - v)^2 = v - 2\hat{e}.v - v^2$$

$$(e+\sigma)^2 = e^2 + \boxed{2e\sigma + \sigma^2}$$

$$= e + 2e\sigma + \sigma^2$$