Braided categories

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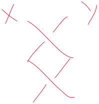

Definition

- *A* **twist** *(***balancing transformation***) on $\mathcal{C}$ is a*

  $$\theta \in \mathrm{Aut}(\mathbf{id}_{\mathcal{C}}) \quad \textit{such that} \quad \theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ \overbrace{c_{Y,X} \circ c_{X,Y}},$$

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- *$\mathcal{C}$ is a* **ribbon tensor category** *if it is rigid and is equipped with a ribbon structure.*

Remark + Example

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Ribbon structure is a non-commutative generalization of a quadratic form.

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If a finite abelian group G has a bilinear form

$$b : G \times G \to \Bbbk^*,$$

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If a finite abelian group G has a bilinear form

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The corresponding quadratic form

$$\theta_{\delta_x} = b(x,x)\mathbf{id}_{\delta_x}, \quad x \in G,$$

defines a ribbon structure on $\mathbf{Vec}_G$.

Connection to the Drinfeld morphism

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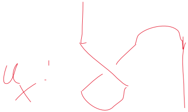

Recall (Section 8.9):

Definition

The **Drinfeld morphism** u *is the natural transformation* $u_X : X \to X^{**}$ *defined as the composition*

$$X \xrightarrow{\mathbf{id}_X \otimes \mathbf{coev}_{X^*}} X \otimes X^* \otimes X^{**} \xrightarrow{c_{X,X^*} \otimes \mathbf{id}_{X^{**}}} X^* \otimes X \otimes X^{**} \xrightarrow{\mathbf{ev}_X \otimes \mathbf{id}_{X^{**}}} X^{**}$$

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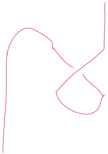
Theorem

If $\mathcal{C}$ *is a braided tensor cat, then* $u_X : X \to X^{**}$ *is an isomorphism*

Idea: define an inverse

v_X :



Proof

WLOG: X is simple

If $\mathcal{C}$ is a braided tensor cat, then $u_X : X \to X^{**}$ is an isomorphism

$v_X \circ u_X = \dots$

$= \underbrace{(ev_{X^*} \circ c^{-1}_{X^*, X^{**}} \circ coev_{X^*})}_{f'} \otimes id_X \otimes \underbrace{ev_X \circ c_{X,X^*} \circ coev_X}_{f}$

To Proof: $f, f' \in \mathrm{End}(\mathbb{1})$ are nonzero

Proof

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Lemma

For any nonzero simple object X the composition

$$f := \mathbf{ev}_X \circ c_{X,X^*} \circ \mathbf{coev}_X \in \mathrm{End}_{\mathcal{C}}(\mathbf{1}) = \mathbb{K}$$

is nonzero.

X simple $\Rightarrow$ f consists of nonzero maps between 1-dim spaces

Corollaries

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Therefore:

Corollary

ψ is a pivotal structure on $\mathcal{C}$ if and only if θ is a twist on $\mathcal{C}$.

Corollaries cont.

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Proposition

Let θ be a twist on $\mathcal{C}$ and $\psi = u \circ \theta$ the canonical pivotal structure. ψ is spherical if and only if θ is a ribbon structure.

Proof uses morphism δ from §7.19 – §7.21

Trace and dimension

Recall (Section 4.7): For $f \in \mathrm{End}_{\mathcal{C}}(X)$ we have

$$\mathbf{Tr}^L(f) : \mathbf{1} \xrightarrow{\mathbf{coev}_X} X \otimes X^* \xrightarrow{f \otimes \mathbf{id}_{X^*}} X^{**} \otimes X^* \xrightarrow{\mathbf{ev}_{X^*}} \mathbf{1},$$
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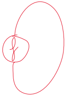

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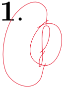

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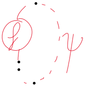

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- $\dim(X)$ takes values in $\Bbbk$ while FP-dim(X) takes values in $\mathbb{R}$.

Proposition

Let $\mathcal{C}$ be a ribbon tensor category with twist θ, then

$$\dim(X) = \mathbf{1} \xrightarrow{\mathbf{coev}_X} X \otimes X^* \xrightarrow{\theta_X \otimes \mathbf{id}_{X^*}} X \otimes X^* \xrightarrow{c_{X,X^*}} X^* \otimes X \xrightarrow{\mathbf{ev}_X} \mathbf{1},$$

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Proof.

$$\begin{tikzcd}
X\otimes X^* \arrow[r, "\mathrm{coev}_{X^*}"] & X\otimes X^*\otimes X^{**}\otimes X^* \arrow[r, "c_{X,X^*}"] \arrow[d, "\mathrm{ev}_{X^*}"] & X^*\otimes X\otimes X^{**}\otimes X^* \arrow[r, "\mathrm{ev}_X"] \arrow[d, "\mathrm{ev}_{X^*}"] & X^{**}\otimes X^* \arrow[d, "\mathrm{ev}_{X^*}"] \\
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\end{tikzcd}$$

$$\hookrightarrow = \mathrm{Tr}^L(u_X) = \theta_X^{-1}\,\mathrm{Tr}(\theta_X u_X) = \theta_X^{-1}\dim(X)$$

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$\square$

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\end{tikzcd}$$

Corollary (Exercise)

If X is simple, then

$$\theta_X^{-1} \dim(X) = \mathbf{Tr}(c_{X,X}^{-1}).$$

8.11. Ribbon Hopf algebras

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Definition

A **Ribbon Hopf algebra** *is a triple* (H, R, v) *such that*

- (H, R) *is a quasitriangular Hopf algebra,*
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Recall:

Definition

A **quasitriangular Hopf algebra** *is a pair* (H, R) *such that*

- H *is a Hopf algebra,*
- $R \in H \otimes H$ *is the universal* R*-matrix of* H*, i.e.,*
  R *is an invertible element satisfying*
  $(\Delta \otimes \mathrm{id})(R) = R^{13}R^{23}$, $(\mathrm{id} \otimes \Delta)(R) = R^{13}R^{12}$, $\Delta^{\mathrm{op}}(h) = R\Delta(h)R^{-1}$, $h \in H$

Main properties

Let (H, R, v) be a ribbon Hopf algebra.

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There is a bijective correspondence between the following:

- Isomorphism classes of ribbon structures on a quasitriangular Hopf algebra (H, R),
- Equivalence classes of ribbon structures on the braided tensor category $\mathrm{Rep}(H)$.
    - The braiding is the one given by R.

Examples

(i) Recall:
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(iv) $u_q(\mathfrak{sl}_2)$, for q a root of unity of odd order, is a ribbon Hopf algebra.

8.12. Characterization of Morita equivalence

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Definition (7.12.17)

Let $\mathcal{C}$ and $\mathcal{D}$ be two tensor cats. $\mathcal{C}$ and $\mathcal{D}$ are **Morita equivalent** *if there is an exact $\mathcal{C}$-module category $\mathcal{M}$ and a tensor equivalence $\mathcal{D}^{\mathrm{op}} \cong \mathcal{C}^*_{\mathcal{M}}$.*

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Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be two finite tensor cats.
C and D are Morita equivalent if and only if $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{D})$ are equivalent as braided tensor cats.

"$\Rightarrow$" direction seen in Prop 8.5.3

This section: "$\Leftarrow$" direction

8.13. The S-matrix of a pre-modular category

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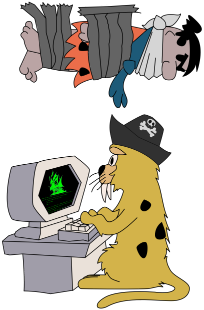

Definition

A **pre-modular category** *is:*

- *a ribbon fusion category,*
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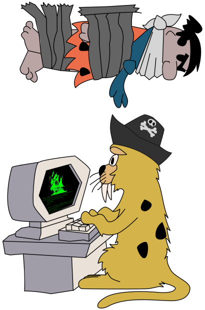

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Definition (S-matrix)

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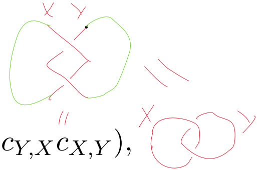

$$S := (s_{XY})_{X,Y\in\mathcal{O}(\mathcal{C})}, \quad \textit{where} \quad s_{XY} = \mathbf{Tr}(c_{Y,X}c_{X,Y}),$$

with $\mathcal{O}(\mathcal{C})$ the set of (isomorphism classes of) simple objects of $\mathcal{C}$.

Remarks

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Definition

A **Modular category** *is a pre-modular category with a non-degenerate S-matrix.*

Example 1

Suppose:

- G is a afinite abelian group,
- $q : G \to \Bbbk^\times$ is a quadratic form on G,
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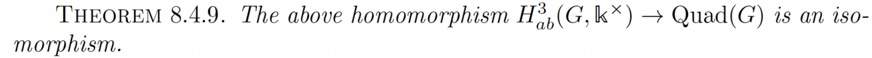

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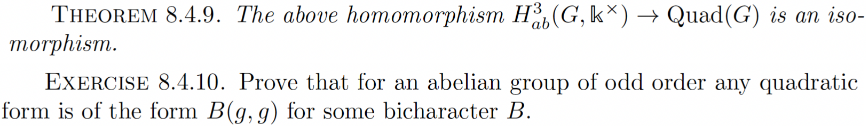

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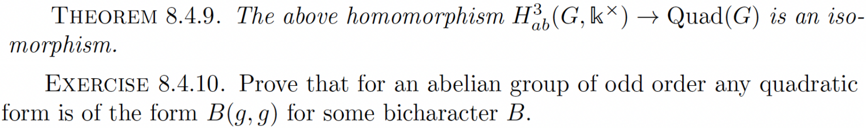

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Corollary

For all pre-metric groups (G, q) there exists a unique up to a braided equivalence pointed braided fusion category $\mathcal{C}(G, q)$ such that the group of isomorphism classes of simple objects is G and the associated quadratic form is q.

Example 1 Continued

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Then:

- $\mathcal{C}(G, q)$ is a pre-modular cat with S-matrix $\{b(g, h)\}_{g,h \in G}$.

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Then:

- $\mathcal{C}(G, q)$ is a pre-modular cat with S-matrix $\{b(g, h)\}_{g,h \in G}$.
- $\mathcal{C}(G, q)$ is a modular cat if and only if q is non-degenerate.

Example 2

Let G be a finite group and $\mathbf{Vec}_G$ the category of G-graded VS.

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Simple objects of $\mathcal{Z}(\mathbf{Vec}_G)$ are parametrized by pairs (C, V), with

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$\mathcal{Z}(\mathbf{Vec}_G)$ is a (pre-)modular fusion cat with twist

$$\theta_{(C,V)} = \frac{\mathbf{Tr}_V(a)}{\dim_{\Bbbk}(V)},$$

and S-matrix given by

$$s_{(C,V),(C',V')} = \frac{|G|}{|C_G(a)||C_G(a')|} \sum_{g \in G(a,a')} \mathbf{Tr}_V(ga'g^{-1})\mathbf{Tr}_{V'}(g^{-1}ag),$$

where $a \in C$, $a' \in C'$, $G(a, a') = \{g \in G | aga'g^{-1} = ga'g^{-1}a\}$.

Properties

Let $\mathcal{C}$ be a pre-modular cat and $X, Y, Z \in \mathcal{O}(\mathcal{C})$.
Denote by $N_{XY}^{Z} := [X \otimes Y : Z]$ the multiplicity of Z in $X \otimes Y$.

In $K_0(\mathcal{C})$: $XY = \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^{Z} Z$ = fusion rules

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Proposition

$$s_{XY} = \theta_X^{-1}\theta_Y^{-1} \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^{Z} \theta_Z \dim(Z)$$

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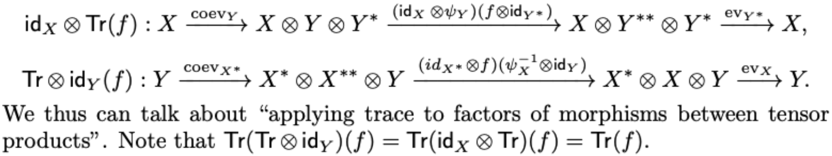
The proof uses that for all $f : X \otimes Y \to X \otimes Y$ we have



$$\mathsf{id}_X \otimes \mathsf{Tr}(f) : X \xrightarrow{\mathrm{coev}_Y} X \otimes Y \otimes Y^* \xrightarrow{(\mathsf{id}_X \otimes \psi_Y)(f \otimes \mathsf{id}_{Y^*})} X \otimes Y^{**} \otimes Y^* \xrightarrow{\mathrm{ev}_{Y^*}} X,$$

$$\mathsf{Tr} \otimes \mathsf{id}_Y(f) : Y \xrightarrow{\mathrm{coev}_{X^*}} X^* \otimes X^{**} \otimes Y \xrightarrow{(id_{X^*} \otimes f)(\psi_X^{-1} \otimes \mathsf{id}_Y)} X^* \otimes X \otimes Y \xrightarrow{\mathrm{ev}_X} Y.$$

We thus can talk about "applying trace to factors of morphisms between tensor products". Note that $\mathsf{Tr}(\mathsf{Tr} \otimes \mathsf{id}_Y)(f) = \mathsf{Tr}(\mathsf{id}_X \otimes \mathsf{Tr})(f) = \mathsf{Tr}(f)$.

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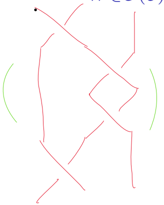
Tr

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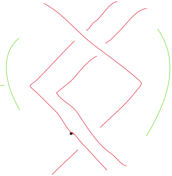

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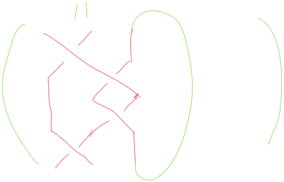

= ???

$\mathrm{Tr}\left(\dim(X)^{-1} s_{XZ}\, c_{YX} c_{XY}\right) = \dim(X)^{-1} s_{XY} s_{XZ}$

Propositions

(i) $\mathcal{O}(\mathcal{C})$ gives rise to characters of the Grothendieck ring $K_0(\mathcal{C})$, i.e.,
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(ii) The numbers $\dfrac{s_{XY}}{\dim(X)}$ are algebraic integers.

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Let $E = \{E_{XY}\}_{X,Y \in \mathcal{O}(\mathcal{C})}$ be the matrix such that

$$E_{XY} = \delta_{X,Y^*} = \begin{cases} 1 & \text{if } X = Y^* \\ 0 & \text{otherwise} \end{cases}$$

Proposition

Let $\mathcal{C}$ be a modular cat and S its S-matrix. Then

$$S^2 = \dim(\mathcal{C})E \quad \text{and} \quad S^{-1} = \{\dim(\mathcal{C})^{-1} s_{XY^*}\}.$$

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Lemma

Let A be a fusion ring with $\mathbb{Z}_+$-basis B, and let χ_1, χ_2 be distinct characters $A \to \Bbbk$. Then

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Corollaries

Corollary (Verlinde formula)

For all $Y, Z, W \in \mathcal{O}(\mathcal{C})$ we have

$$\sum_{X \in \mathcal{O}(\mathcal{C})} \frac{s_{XY} s_{XZ} s_{XW^*}}{\dim(X)} = \dim(\mathcal{C}) N_{YZ}^{W},$$

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For all $Z \in \mathcal{O}(\mathcal{C})$ we define the square matrices

$$D^Z := \left(\delta_{X,Y} \frac{s_{XZ}}{\dim(X)} \right)_{X,Y \in \mathcal{O}(\mathcal{C})}, \quad \text{and} \quad N^Z := (N_{XY}^{Z})_{X,Y \in \mathcal{O}(\mathcal{C})}.$$

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Corollary

Conjugation by the S-matrix diagonalizes the fusion rules of $\mathcal{C}$, i.e.,

$$D^Z = S^{-1} N^Z S, \quad \textit{for all } Z \in \mathcal{O}(\mathcal{C}).$$

Proposition

Let C be a modular category and $X \in \mathcal{O}(\mathcal{C})$. Then

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is an algebraic integer.

Theorem (entries of S lie in a cyclotomic field)

$\Bbbk = \mathbb{C}$

There exists a root of unity $\xi \in \Bbbk$ such that $s_{XY} \in \mathbb{Q}(\xi)$.