Kleine Seminar — Chapter 8

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0 A motivating example

Consider the braid group B_n given by the generators

$$\{\sigma_1, \dots, \sigma_{n-1}\}$$

and the relations

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}$$

braid rel

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i-j|>1$$

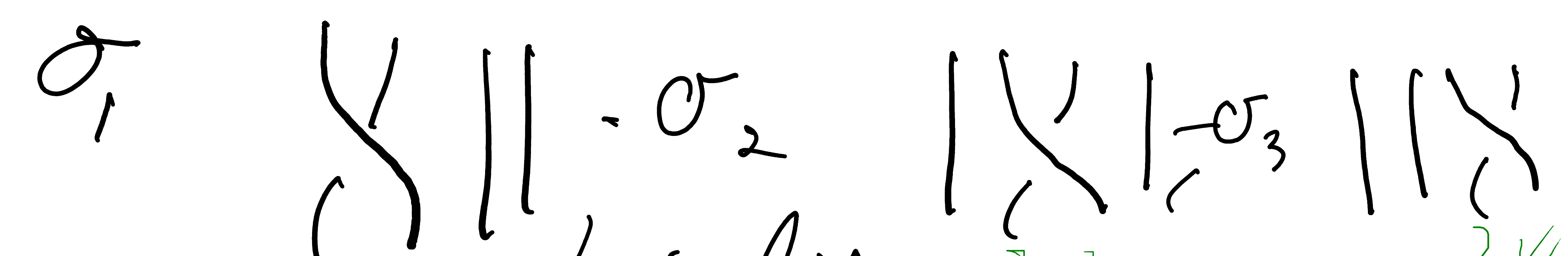
S_n + $\sigma_j^2 = 1$

$|B_n| = \infty$

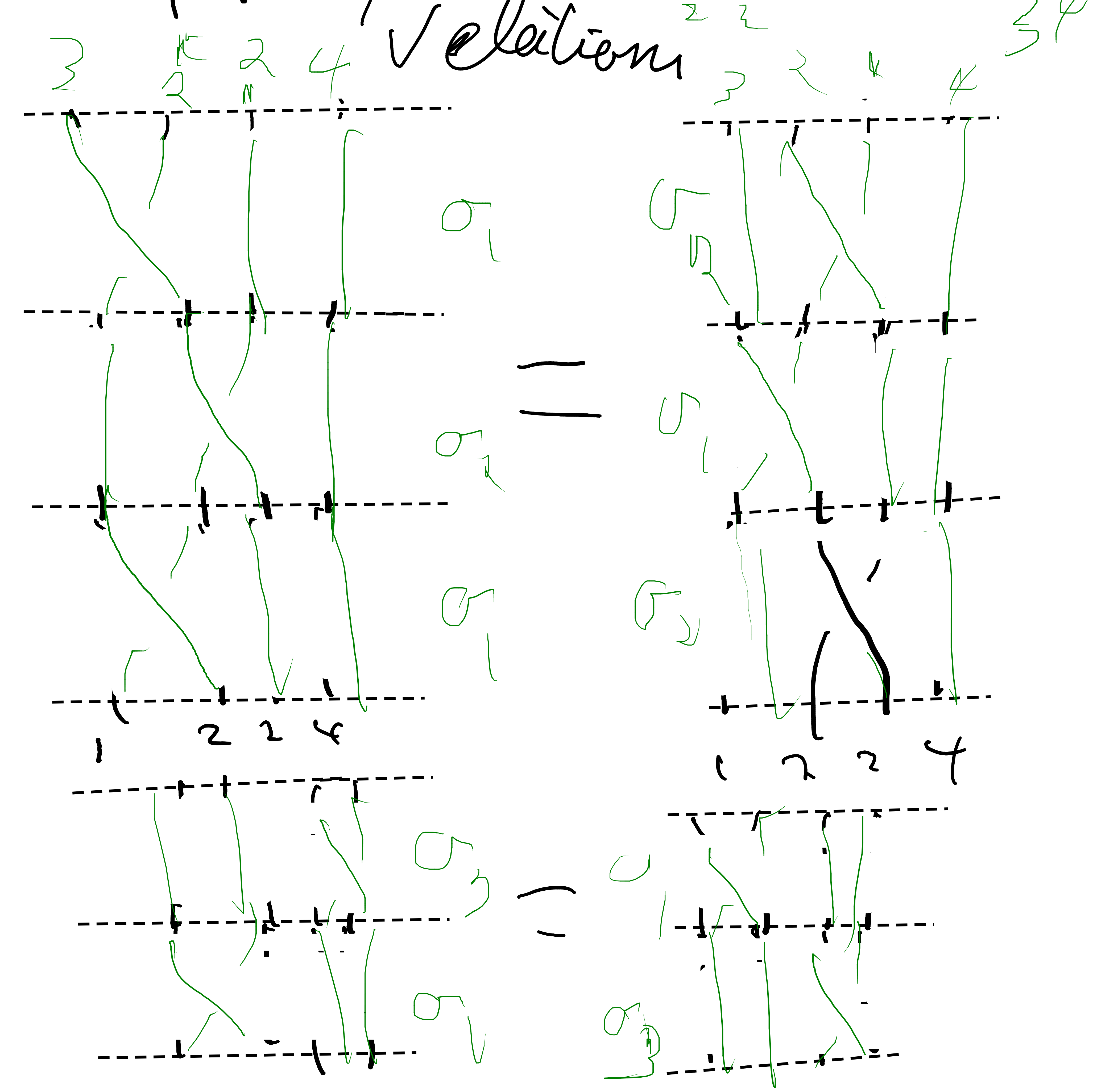
$B_n \twoheadrightarrow S_n$

$|S_n| = n!$

Diagrammatic $n=4$



relations



We want to categorify this group.

Consider the <u>braid category</u> $\mathcal{B}$

It is a monoidal category generated by

obj: $\underline{1}$, ($\mathbb{N}$ since $\otimes$: $\underline{1}\otimes\underline{1} = \underline{2}$)

Morp: id: $\underline{1} \to \underline{1}$

b : $\underline{2} \to \underline{2}$

and relations:

$$\mathrm{id}\otimes b \circ b\otimes \mathrm{id} \circ \mathrm{id}\otimes b = b\otimes \mathrm{id} \circ \mathrm{id}\otimes b \circ b\otimes \mathrm{id}$$

and

$$b \circ \mathrm{id} \circ \mathrm{id} = \mathrm{id} \circ \mathrm{id} \circ b$$

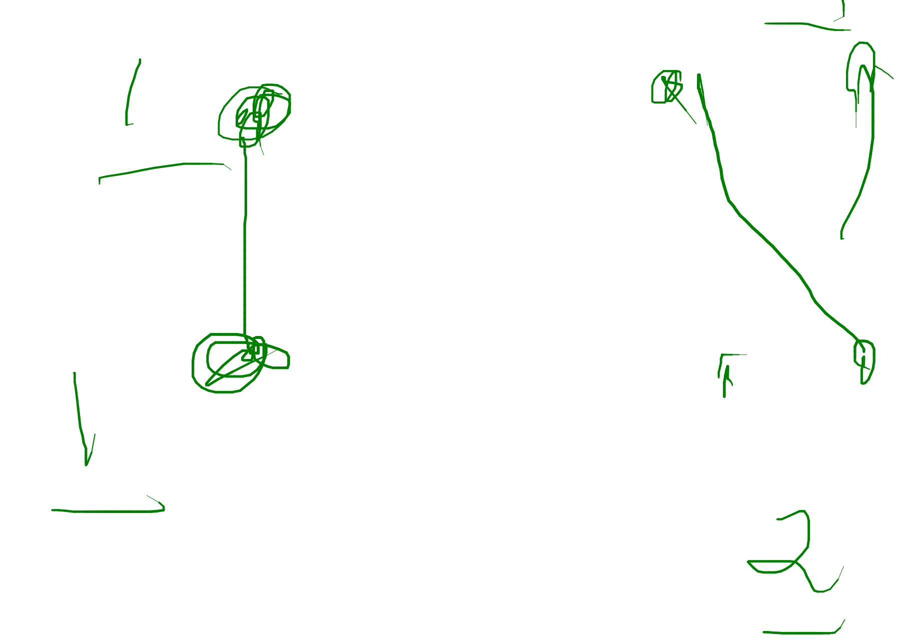
$$b \circ \mathrm{id} \circ \mathrm{id} \circ \mathrm{id} = \mathrm{id} \circ \mathrm{id} \circ \mathrm{id} \circ b$$

interchange

Obj: $\underbrace{\circ \quad \circ \quad \circ}_{\underline{n}}$

$\circ \otimes \circ = \circ\ \circ$

Mor



$| \otimes | = | \; |$

Braid

$\mathrm{Hom}(\underline{n}, \underline{m}) = B_n$ if $n = m$, $\emptyset$ else

Goal:

① Define a category following these properties

② Show a coherence theorem: "You need only to look at braids!"

③ Links with Hopf algebras

④ premetric group ⟶ fusion braid cat

⑤ Center of Cat

Braided cats

monocat

braidedcat

braiding

1 Braided category

coherence properties.

DEFINITION 8.1.1. A *braiding* (or a *commutativity constraint*) on a monoidal category $\mathcal{C}$ is a natural isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ such that the hexagonal diagrams

(8.1)

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[r, "c_{X,Y \otimes Z}"] & (Y \otimes Z) \otimes X \arrow[dr, "a_{Y,Z,X}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "a_{X,Y,Z}"] \arrow[dr, "c_{X,Y} \otimes \mathrm{id}_Z"'] & & & Y \otimes (Z \otimes X) \\
 & (Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes c_{X,Z}"'] &
\end{tikzcd}$$

and

(8.2)

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[r, "c_{X \otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[dr, "a^{-1}_{Z,X,Y}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "a^{-1}_{X,Y,Z}"] \arrow[dr, "\mathrm{id}_X \otimes c_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \\
 & X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"'] & (X \otimes Z) \otimes Y \arrow[ur, "c_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd}$$

commute for all objects X, Y, Z in $\mathcal{C}$.

DEFINITION 8.1.2. A *braided* monoidal category is a pair consisting of a monoidal category and a braiding.

a: associator

c : braiding

if $\mathcal{C}$ is strict monocat

$$\begin{tikzcd}
X \otimes Y X Z \arrow[rr, "c_{X,YZ}"] \arrow[dr, "c_{XY}"'] & & Y \otimes Z X \\
 & Y X Z \arrow[ur, "c_{XZ}"'] &
\end{tikzcd}$$

in braids

$n \otimes m \to m \otimes n$

Def $c^{-1}_{X,Y}$: the inverse of $c_{X,Y}$, define the inverse braided category of $(\mathcal{C}, c)$

$\mathcal{C}, c$ $\quad$ $\mathcal{C}^{rev}$ for c^{-1}

Def Let $(\mathcal{C}_1, c^1)$ and $(\mathcal{C}_2, c^2)$ be two braided categories, a braided functor F, J is a monoidal functor with

$$\begin{tikzcd}
F(X)\otimes F(Y) \arrow[r, "c^2_{F(X),F(Y)}"] \arrow[d, "J_{X,Y}"'] & F(Y)\otimes F(X) \arrow[d, "J_{Y,X}"] \\
F(X\otimes Y) \arrow[r, "F(c^1_{X,Y})"'] & F(Y\otimes X)
\end{tikzcd}$$

monoidal functor: add the structure J.

braided monoidal functor: a property.

Yang-Baxter equation:

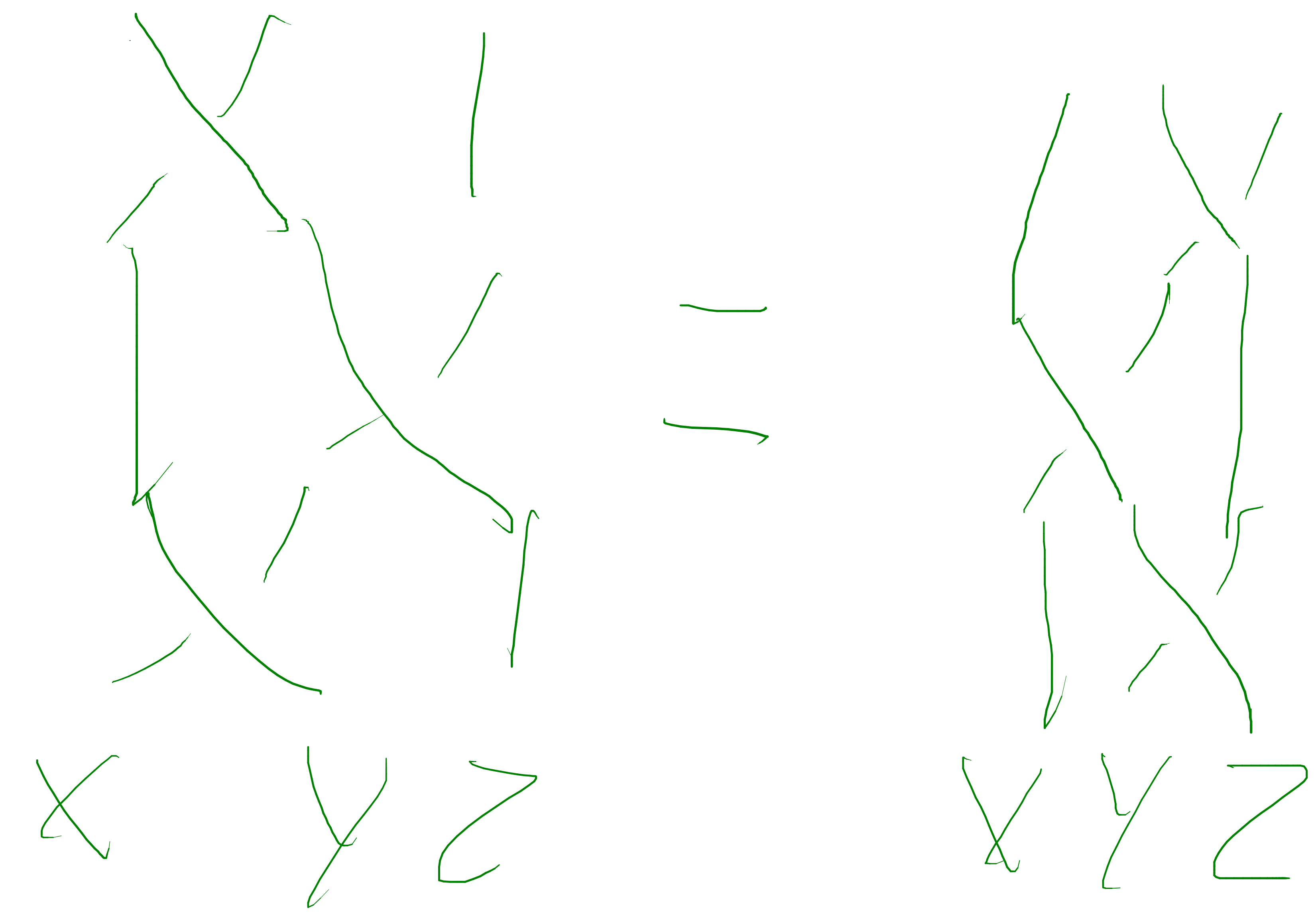
$\mathcal{C}$ strict monoidal category with braiding c. Then

$$(c_{YZ}\otimes id_X)\circ(id_Y\otimes c_{XZ})\circ(c_{XY}\otimes id_Z)$$
$$\|$$
$$(id_Z\otimes c_{XY})\circ(c_{XZ}\otimes id_Y)\circ(id_X\otimes c_{YZ})$$

Proof:

$$\begin{tikzcd}
 & X\otimes Y\otimes Z \arrow[rr, "c_{X,Y}\otimes \mathrm{id}_Z"] \arrow[dl, "\mathrm{id}_X\otimes c_{Y,Z}"'] \arrow[dd, "c_{X\otimes Y,Z}"] & & Y\otimes X\otimes Z \arrow[dr, "\mathrm{id}_Y\otimes c_{X,Z}"] \arrow[dd, "c_{Y\otimes X,Z}"'] & \\
X\otimes Z\otimes Y \arrow[dr, "c_{X,Z}\otimes \mathrm{id}_Y"'] & & & & Y\otimes Z\otimes X \arrow[dl, "c_{Y,Z}\otimes \mathrm{id}_X"] \\
 & Z\otimes X\otimes Y \arrow[rr, "\mathrm{id}_Z\otimes c_{X,Y}"'] & & Z\otimes Y\otimes X. &
\end{tikzcd}$$

Braid



A braided category is <u>symmetric</u> if $c_{YX} \circ c_{XY} = id_{X \otimes Y}$

Examples of symmetric braided cat

(1) Vec, Set, Rep G

braid: transposition of factors

(2) $\mathbb{K}$ with $\operatorname{char} \mathbb{K} \neq 2$.

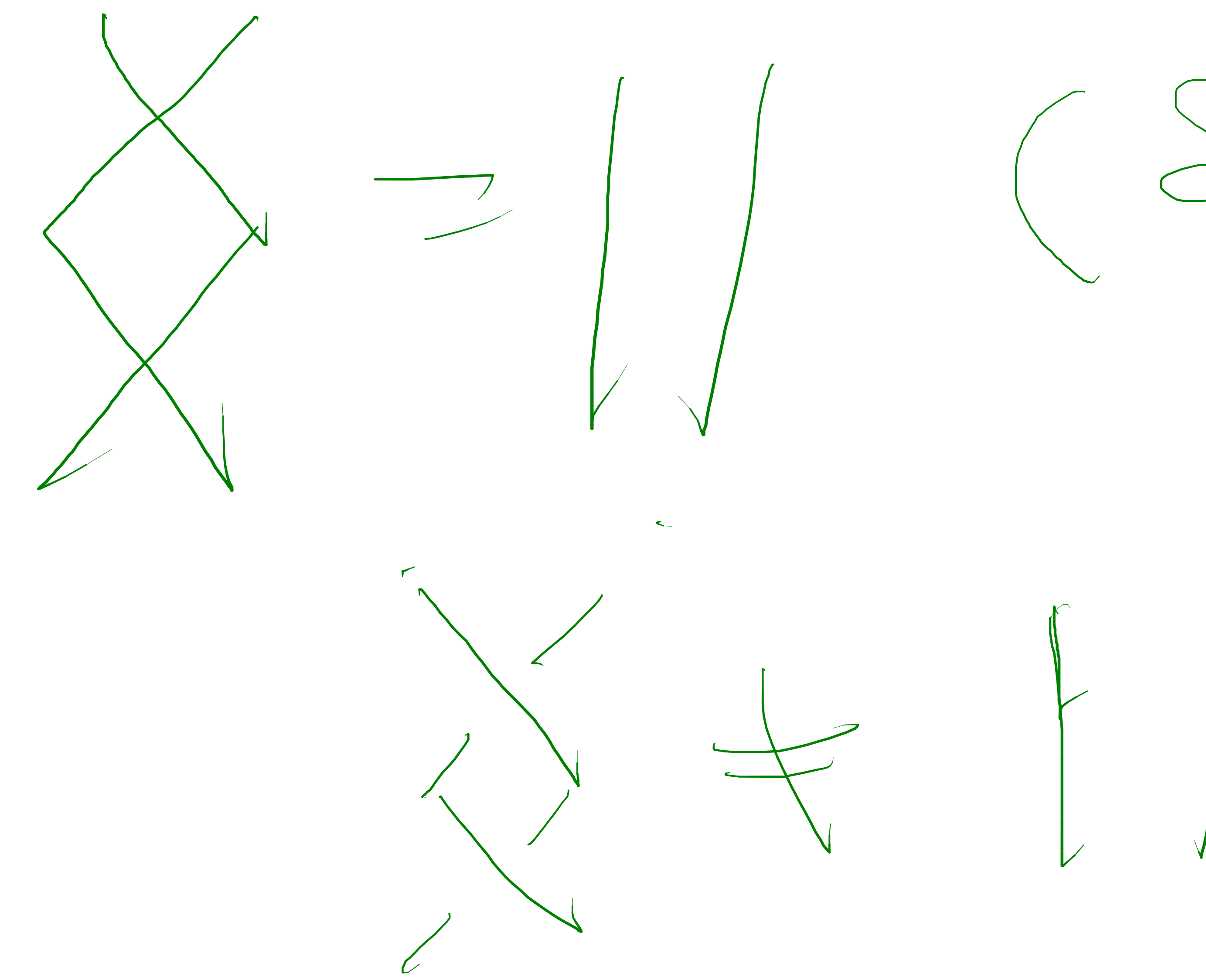
$Vec_{\mathbb{Z}/2\mathbb{Z}}$ is braided:

$$c_{X,Y}(x \otimes y) = (-1)^{\deg x \deg y} y \otimes x$$

sVec

x, y homogeneous

$c_{YX} \circ c_{XY} = id$



(S_n)

Yang-Baxter give in fact a group homo

$$B_n \to \mathrm{Aut}_{\mathcal{C}}(V^{\otimes n})$$

for every object V in $\mathcal{C}$, strict monoidal braided cat.

$$\sigma_i \mapsto \mathrm{id}_{V^{\otimes(i-1)}} \otimes c_{VV} \otimes \mathrm{id}_{V^{\otimes(n-i-1)}}$$

Remark

Mac Lane coherence thm on monocats: we can consider strict monocat

EXERCISE 8.2.7. Let $\mathcal{C}$ be a braided tensor category (not necessarily strict), and let $X_1, \ldots, X_n \in \mathcal{C}$. Let P_1, P_2 be any parenthesized products of $X_1, \ldots, X_n$ (in any orders) with arbitrary insertions of unit objects $\mathbf{1}$. Let $f = f_C : P_1 \to P_2$ be an isomorphism, obtained as a composition C of associativity, braiding, and unit isomorphisms and their inverses possibly tensored with identity morphisms. Explain how C defines a braid b_C. Show that if $b_C = b_{C'}$ in B_n then $f_C = f_{C'}$. This statement is called ~~Mac Lane's~~ *braided coherence theorem.*

Joyal 'n Street

Step 1: go for strict by Mac Lane

((monocat

Step 2:

$$\begin{tikzcd} F : \mathcal{B} \arrow[r] \arrow[d] & \mathcal{C} \\ a\ \mathcal{C}_0 \end{tikzcd}$$

$\mathcal{C}_0$ is category forget braiding

Exercise 8.2.7

set of M

$$\mathrm{Hom}_{BMC}(B, M) \xrightarrow{\ \phi\ } M_0$$

$M = C$

functors of $B \to M$ braid

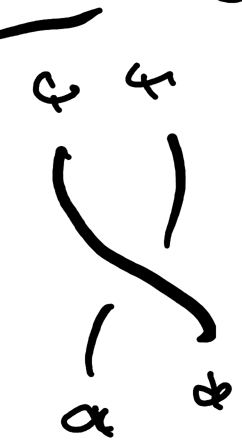
ϕ evaluate at 1

1): We send each functor F to $F(1)$.

2) For each $a \in M_0$ we want to find a functor with $F(1) = a$

i) $\otimes$ must be preserved so

$$F(n) = a^n$$

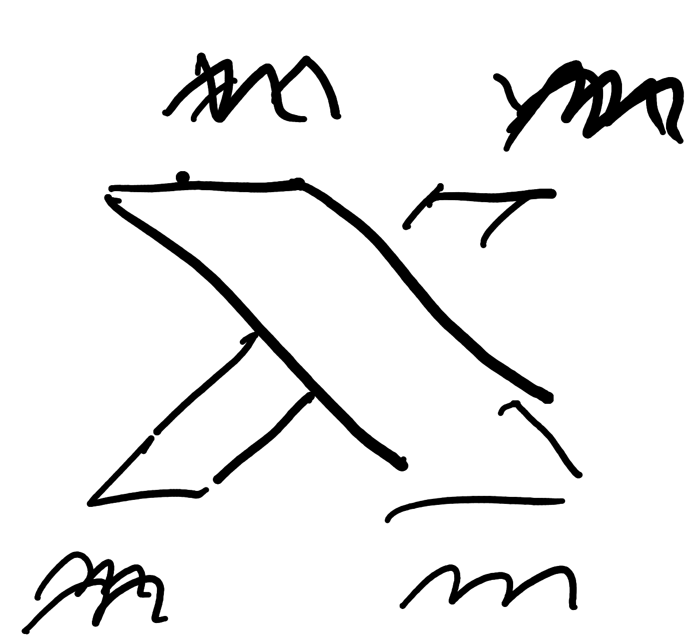
ii) for σ_i:  we must send it to

$$F(\sigma_i) = c_{a,a}$$

and more generally

$$F(\sigma_{nm}) = c_{nm}$$

$F(\sigma_1) \to c_{aa}$

$\sigma_{XY} =$ 

So, for $\sigma_i = \mathrm{id}_{i-1} \otimes \sigma_1 \otimes \mathrm{id}_{n-i-1}$

$$F(\sigma_i) = 1_{i-1} + c_{a,a} + 1_{n-i}$$

2ii) Check this functor preserve the relation of B (Yang Baxter)

iv) find a map

$\to J(m,n) = Fm \otimes Fn \to F_{m+n}$

So $a^m \cdot a^n = a^{m+n}$

So, for braid, identity

Now, prove "evaluat at 1" is an equivalence of cat.

i) faithful ✓

ii) full

Given F, G functors and $F \xrightarrow{\delta} G$:

$$\begin{array}{ccc} F(1)^n & \xrightarrow{\delta^n} & G(1)^n \\ F_n \downarrow \wr & & \wr \downarrow G_n \\ F(n) & \xrightarrow[\theta_n]{\sim} & G(n) \end{array}$$

F_n and G_n are from the monoidal structure

$F(1)^2 = F(1)F(1) \underset{\exists}{\cong} F(1 \otimes 1) = F(2)$ ✓

and they are iso.

by inductn

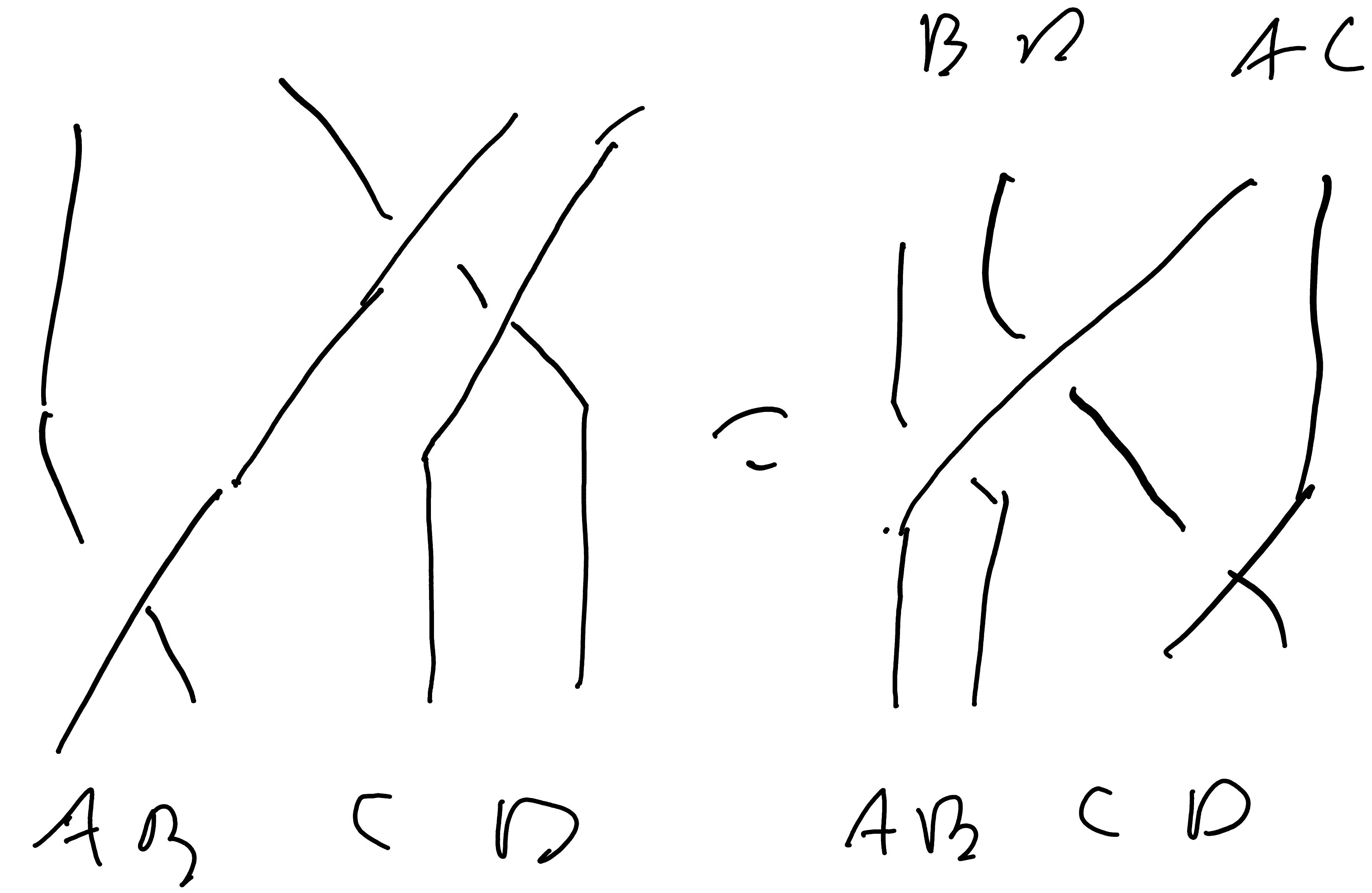
then $\theta_n = G_n \delta_n F_n^{-1}$

and we have the equivalence of categories.

So if $\delta_{c_1}: P_1 \to P_2$ and $\delta_{c_2}: P_1 \to P_2$ are linked by the same braid, then they are equal.

The importance of this result is that it enables to use the intuition of braids when looking at the n fold tensor product in a braided monoidal category and that simplify some proofs

Ex: (Joyal Street)



$$\begin{tikzcd}
ABCD \arrow[r, "c11"] \arrow[d, "11c"'] & BACD \arrow[dr, "1c1"] & \\
ABDC \arrow[r] & BDAC \arrow[r, squiggly] & BDAC
\end{tikzcd}$$

instead of working with associator

Part on Hopf algebras..

→ Goal is to introduce a generalisation of cocommutative Hopf algebras and give braiding there

Reminder on Hopf algebras

- bialgebra $H, \mu, i, \Delta, \varepsilon, S$

$\mu : H \otimes H \to H$

$i : K \to H$

$\Delta : H \to H \otimes H$

$\varepsilon : H \to K$

$S : H \to H$

$$\mu \circ (id \otimes S) \circ \Delta = i \circ \varepsilon = \mu \circ (S \otimes id) \circ \Delta \quad \text{*}$$

8.3 Quasitriangular Hopf algebras

Def H Hopf algebra and $R \in H \otimes H$ satisfying

$\Delta \otimes id\, R = R^{13} R^{23}$

$id \otimes \Delta R = R^{13} R^{12}$

$\Delta^{op}_h = \sigma \circ \Delta(h) = R \Delta(h) R^{-1} \quad \forall h \in H$

$\Delta^{op} h = \sigma \circ \Delta h = R \Delta h R^{-1}$

permutation

called the universal R-matrix.

We have quantum YB

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}$$

(H, R) is a quasitriangular Hopf algebra

Suppose $\mathrm{Rep}\, H$ $(= \mathcal{C})$ is braided with braiding c_{XY}.

Define $c^{\vee}_{H} = \sigma \circ c_{HH} : H \otimes H \to H \otimes H$ with σ permutation of component.

$c^{\vee}_{H,H}$ commute with right multiplication so it is determined by a $R \in H \otimes H$ invertible.

$R^{12} = a' \otimes b' \otimes 1$ if $R = a' \otimes b'$

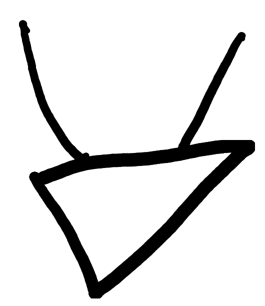
$R = a' \otimes b'$

$R^{23} = 1 \otimes a' \otimes b'$

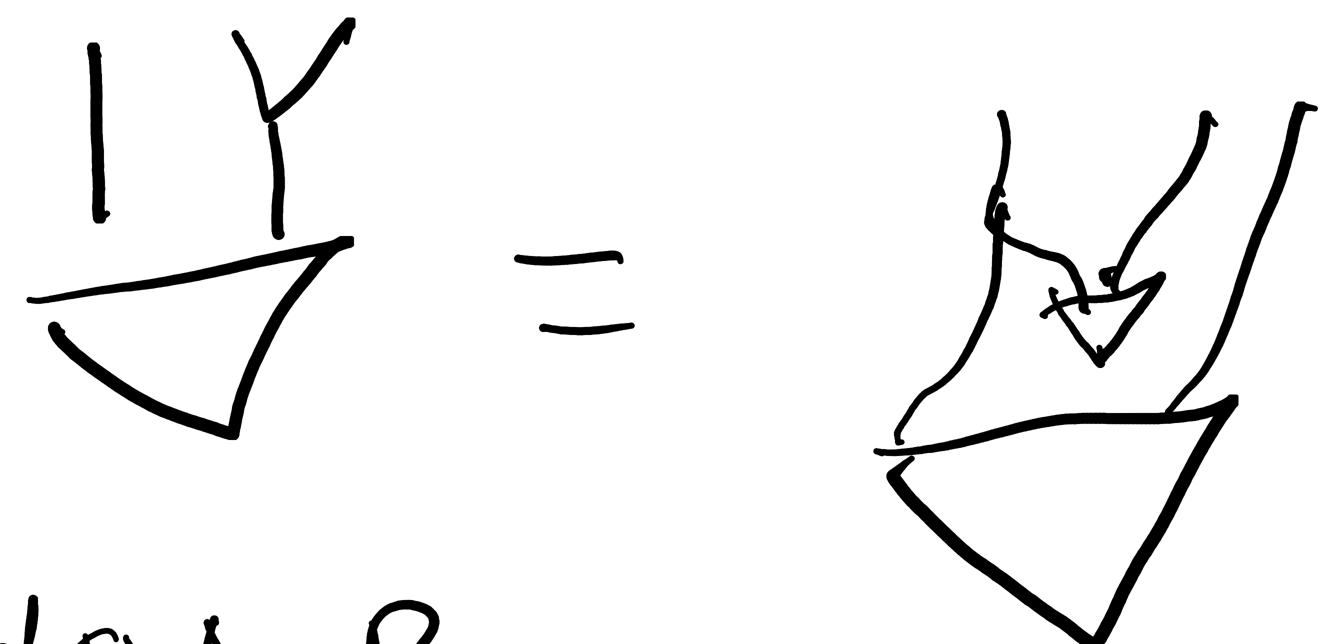
$R^{13} \quad a' \otimes 1 \otimes b'$

In pictograms

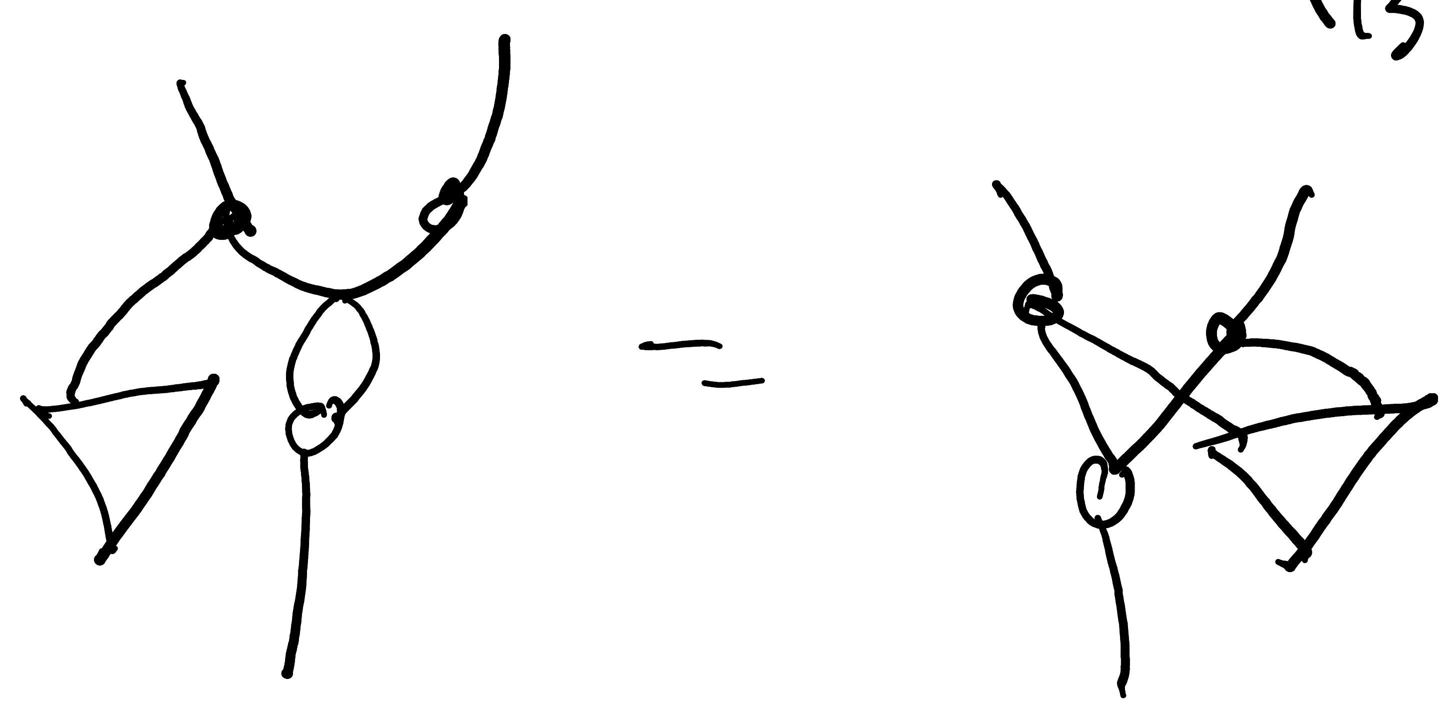
— MSc thesis Giovanni de Felice

$R =$ 

$(\Delta \otimes id) \circ R = R_{13} \cdot R_{23}$

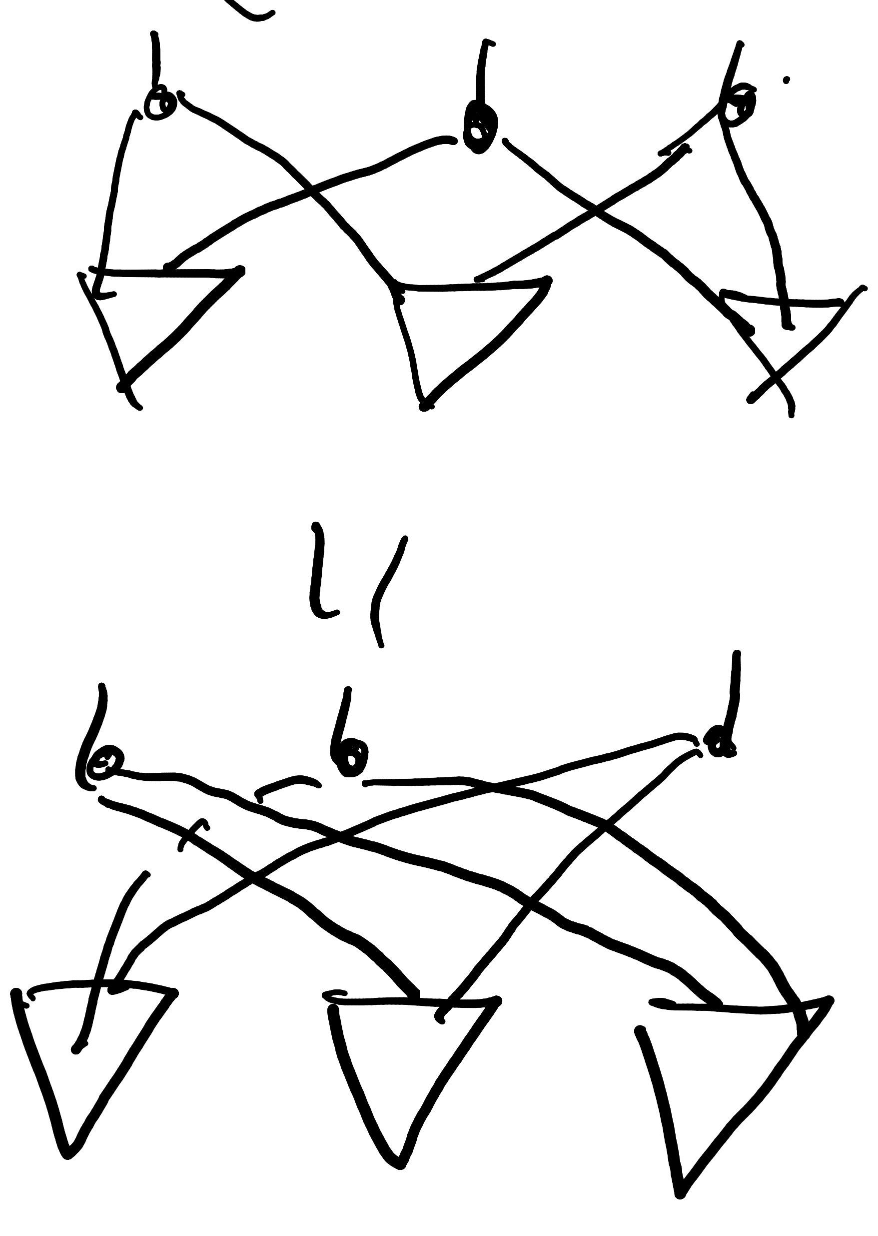


$id \otimes \Delta \circ R = R_{13} \cdot R_{12}$



$\Delta^{op} \cdot R = R \cdot \Delta$

Quantum YB



$R_{12} R_{13} R_{23}$

$=$

$R_{23} R_{13} R_{12}$

Def If H, R is a quasitriangular Hopf algebra and $R^{-1} = R^{21}$ then R is called unitary and (H, R) is called a triangular Hopf algebra

Prop (Drinfeld double)
$D(H) = H \otimes H^{*cop}$, for H fin-dim Hopf algebra, is a quasi-triangular Hopf algebra with $R = \sum h_i \otimes h_i^*$

with multiplication the unique extension of mult. in H and H^{*cop} making R invertible & R-matrix

basis H — dual basis H^{*cop}

This happens for example when C is symmetric.

ex: H cocommutative then $R = 1 \otimes 1$ makes it triangular structure

8.3.6 $k\,\mathbb{Z}/2\mathbb{Z}$ $\operatorname{char} k \neq 2$, $\langle g \rangle = \mathbb{Z}/2\mathbb{Z}$

$R = 1 \otimes 1$

$R' = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)$

$R \to$ Vec

$R' \to$ sVec

Prop Let J be a twist, if (H, R) is quasitriangular Hopf algebra, then $(H^J, R^J = (J^{21})^{-1} R J)$ is also and

$$\operatorname{Rep} H \simeq \operatorname{Rep} H^J$$

as braided categories

Co quasitriangular Hopf algebra $\to$ invert the arrows

Thus, a bialgebra twist for H is an invertible element $J \in H \otimes H$ such that $(\varepsilon \otimes \mathrm{id})(J) = (\mathrm{id} \otimes \varepsilon)(J) = 1$, and J satisfies the *twist equation*

$$(\mathrm{id} \otimes \Delta)(J)(\mathrm{id} \otimes J) = (\Delta \otimes \mathrm{id})(J)(J \otimes \mathrm{id}). \tag{5.30}$$

This means that two quasitriangular Hopf algebras can have equivalent categories of modules (braided) while being different.

Coquasitriangular Hopf algebras are duals to quasitriangular Hopf algebras and thus generalize commutative Hopf algebras.

Suppose that (A, R) is a finite dimensional quasitriangular Hopf algebra, and $H = A^*$. Then $R \in A \otimes A$ induces a bilinear form $H \otimes H \to \Bbbk$ (which we will also denote by R), and the properties of $R \in A \otimes A$ can be rewritten in terms of this form. This motivates the following definition.

DEFINITION 8.3.19. A *coquasitriangular* Hopf algebra is a pair (H, R), where H is a Hopf algebra over $\Bbbk$ and $R : H \otimes H \to \Bbbk$ (the *R-form*) is a convolution-invertible bilinear form on H satisfying the following axioms:

$$R(h, lg) = \sum R(h_1, g)R(h_2, l), \;\; R(gh, l) = \sum R(g, l_1)R(h, l_2)$$

and

$$\sum R(h_1, g_1)h_2g_2 = \sum g_1h_1R(h_2, g_2) \;\; (h, g, l \in H).$$

If $\sum R(h_1, g_1)R(g_2, h_2) = \varepsilon(g)\varepsilon(h)$ then (H, R) is called *cotriangular*.

Ex! Commutative Hopf algebras are cotriangular $R = \varepsilon \otimes \varepsilon$

Abelian group + bil. $R: A \times A \to \Bbbk^\times$

$(\Bbbk A, R)$ coquasitriangular

+ R symmetric cotriangular

Example of $U_q(\mathfrak{sl}_2)$

$V_\lambda = \underline{W_\lambda} \otimes \underline{1\text{-d rep}}$ $\begin{cases} E \to 0 \\ F \to 0 \\ K \to \{\pm 1, -1\} \end{cases}$

Type I : $K \to +1$

→ For q not a root of unity. Category of fin-dim rep of type I (Highest weight with $K \to +1$)

$$R = R_0 \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n E^n \otimes F^n}{[n]_q!}$$

$$R_0 = \sum_{i,j\in\mathbb{Z}} q^{ij/2} \mathbb{1}_i \otimes \mathbb{1}_j$$

$\mathbb{1}_j$: proj to weight j

This define a braiding on $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$

8.4 Premetric group and partial braided fusion categories

G abelian group.

Def quadratic form $q : G \to \mathbb{k}^\times$ such that

$$q(g) = q(g^{-1}) \quad (1)$$

and

$$b(g,h) = \frac{q(gh)}{q(g)q(h)} \quad (2)$$

is a bicharacter, so

$$b(g_1 g_2, h) = b(g_1,h)\, b(g_2,h)$$

q is non-degenerate if b is non-degenerate

(G,q) is a premetric group,
metric group if q is non-degenerate

ex. if B is a bichar,
$q(g) = B(g,g)$ is a quadratic form,
furthermore, all forms for $|G|$ odd are like this.

For group of even order this doesn't hold

Let C be a fusion braided category with a $g \in C$ (pointed)

$$C \simeq \mathrm{Vec}_G^{\omega} \text{ as tensor cat}$$

For $g \in G = \{\text{iso class of simple object}\}$

$$q(g) = c_{XX} \in \mathrm{Aut}_C(X \otimes X) = K^*$$

X simple with iso class $= g$.

Lemma $q : G \to K^*$ is a quadratic form

Proof. $q(gh) = q_g q_h \underbrace{b(g,h)}_{c_{YX}\, c_{XY}}$

Then check with braid axiom that it is a bicharacter.

Goal of the section: Proof that

$$F : \text{Pointed braided fusion cat} \longrightarrow \text{Pre metric groups}$$

is an equivalence

$$b_{gh} = \frac{q_{gh}}{q_g q_h}$$

Things I skipped

C_1, C_2 skeletal pointed braided fusion cats described by G_1 G_2 with abelian cocycle $(\omega_1, c_1) \in Z^3_{ab}(G_1, k^\times)$, $(\omega_2, c_2) \in Z^3_{ab}(G_2, k^\times)$

$$
\begin{aligned}
\omega(g_1, g_2, g_3) &= k(g_2, g_3) k(g_1 g_2, g_3)^{-1} k(g_1, g_2 g_3) k(g_1, g_2)^{-1}, \\
c(g_1, g_2) &= k(g_1, g_2) k(g_2, g_1)^{-1}.
\end{aligned}
\tag{8.12}
$$

where $(\omega, c) = (\omega_1, c_1)^{-1} f^*(\omega_2, c_2)$.

For an abelian group G let $B^3_{ab}(G, \Bbbk^\times) \subset Z^3_{ab}(G, \Bbbk^\times)$ be the subgroup of *abelian coboundaries*, that is, of the abelian cocycles defined by (8.12) with $f = \mathsf{id}$ for all functions $k : G \times G \to \Bbbk^\times$.

DEFINITION 8.4.7. The group $H^3_{ab}(G, \Bbbk^\times) := Z^3_{ab}(G, \Bbbk^\times)/B^3_{ab}(G, \Bbbk^\times)$ is called the *abelian cohomology group* of G with coefficients in $\Bbbk^\times$.

Let $\mathrm{Quad}(G)$ be the group of quadratic forms with values in $\Bbbk^\times$ on a finite abelian group G. It is easy to check (and it follows from the discussion above) that the homomorphism $H^3_{ab}(G, \Bbbk^\times) \to \mathrm{Quad}(G)$, $(\omega, c) \mapsto q(g) = c(g, g)$ is well defined. The following result is due to Eilenberg and Mac Lane. For our proof we will need some results which will be proved later.

THEOREM 8.4.9. *The above homomorphism $H^3_{ab}(G, \Bbbk^\times) \to \mathrm{Quad}(G)$ is an isomorphism.*

85 The center on a braided cart

Center as braided category

Back to 7.13

$\hookrightarrow$ $\mathcal{C}$ monoidal category

$$\mathcal{Z}(\mathcal{C}) = \{ (Z, \gamma) \mid \gamma_X : X \otimes Z \xrightarrow{\sim} Z \otimes X \}$$

$\forall X \in \mathcal{C}$

satisfying

$$\begin{tikzcd}
 & X(ZY) \arrow[r, "a"] & (XZ)Y \arrow[dr, "\gamma_X"] & \\
X(YZ) \arrow[ur, "\gamma_Y"] \arrow[dr, "a"'] & & \arrow[l, phantom, "\circlearrowleft"] & (ZX)Y \\
 & (XY)Z \arrow[r, "\gamma_{XY}"] & Z(XY) \arrow[ur, "a"'] &
\end{tikzcd}$$

$\mathcal{Z}(\mathcal{C})$

It is a braided category with

$$c_{(Z,\gamma),(Z',\gamma')} = \gamma'_Z$$

We have

$$\mathcal{Z}(\mathcal{C}^{op}) \simeq \mathcal{Z}(\mathcal{C})^{rev}$$

$|$ as braid cat $\qquad$ $\uparrow$ reverse braiding $\downarrow$

$$\gamma'_Z = (\gamma'^{-1}_Z)$$

Prop $\mathcal{C}$ fin tensor cat

$\mathcal{M}$ indecomposable exact

$\mathcal{C}$-mod cat.

$$Z(\mathcal{C}^*_{\mathcal{M}}) \simeq Z(\mathcal{C})^{rev}$$

as braided tensor cat

$Z(\mathcal{C}^*)$

Marcelo told us

$\exists A$ alg in $\mathcal{C}$ s.t.

$$\mathcal{M} \simeq \mathrm{Mod}_{\mathcal{C}}(A).$$

$$(7.16.3) \quad Z(\mathcal{C}) \xrightarrow{\sim} Z(\mathrm{Bimod}_{\mathcal{C}} A)$$

$$Z \longmapsto Z \otimes A$$

But then, this equivalence respect braiding and

$$\mathcal{C}^{*\,op}_{\mathcal{M}} \simeq \mathrm{Bimod}\, A$$

$$Z(\mathcal{C}^*_{\mathcal{M}}) \overset{so}{\simeq} Z(\mathcal{C}^{op}) \simeq Z(\mathcal{C})^{rev}$$ [illegible]

Finite group

EXAMPLE 8.3.9. Let G be a finite group. Then the underlying algebra of the Drinfeld double $D(G) := D(\Bbbk G)$ of $\Bbbk G$ is the semidirect product $\mathsf{Fun}(G, \Bbbk) \rtimes \Bbbk G$, where G acts on $\mathsf{Fun}(G, \Bbbk)$ by conjugation, and the universal R-matrix is $R = \sum_{g\in G} g \otimes \delta_g$, where δ_g is the delta-function at g.

Center is a quasitriangular Hopf algebra

Ex 8.5.4 Center of Vec_G.

G-graded vector space.

An object in $Z(\mathrm{Vec}_G)$ is

$$V = \bigoplus_{g\in G} V_g \,, \quad \{\gamma_x : \delta_x \otimes V \xrightarrow{\sim} V \otimes \delta_x\}$$

satisfying the hexagon

So we have

$$u_{g,x} : V_{gxg^{-1}} \longrightarrow V_x \qquad g, x \in G$$

from the grading.

Set $u_x : \bigoplus_{g\in G} u_{g,x}$. So $u_x : V \xrightarrow{\sim} V$

V is G-equivariant

So objects of $Z(\mathrm{Vec}_G)$ are identified with G-equivariant G-graded vector spaces

$$\begin{tikzcd}
 & X(ZY) \arrow[r] & (XZ)Y \arrow[dr] & \\
X(YZ) \arrow[ur] \arrow[dr] & & & (ZX)Y \\
 & (XY)Z \arrow[r] & Z(XY) \arrow[ur] &
\end{tikzcd}$$

G equivariant 2.7.2

V is $u = \{u_g : T_g(V) \to V\}$

$$\begin{tikzcd}
T_g(T_h(V)) \arrow[r] \arrow[d, "\gamma"'] & T_g(V) \arrow[d, "u_g"] \\
T_{gh} X \arrow[r, "u_{gh}"] & V
\end{tikzcd}$$

Monoidal arrow

Simple objects of $Z(\mathrm{Vec}_G)$ are in bijection with (C, V)

C: fin. conjugacy class

V irreducible fin-dim rep of $\mathrm{cent}(g)$

If G is finite then $Z(\mathrm{Vec}_G)$ is a fusion cat

Then, the Frobenius Perron dimension of $C(C, V)$ is $|C| \dim_k(V)$

then $\mathrm{FPdim}(Z(\mathrm{Vec}_G)) = |G|^2$

$\mathrm{FP}(C, V) = |C| \dim V$

Remark 4.15.8

X simple, $G_X \subset G$ the stabilizer of iso class of X.

$1 \to k^\times \to \widetilde{G}_X \to G_X \to 1$

$\widetilde{G}_X : (h, T_h(X) \cong X)$

Set of iso classes in bijection with $\mathrm{Irr}'(\widetilde{G}_X)$ of fin-dim irreps.

→ Prop 4.15.9

Y correspond to $V \in G_X$

then $\mathrm{FPdim}\, Y = \dim V$

$X : \underbrace{[G : G_X]}_{= |C|} \mathrm{FPdim}\, Y$

Recap

Braid categories

C, X, Y

overcrossing: