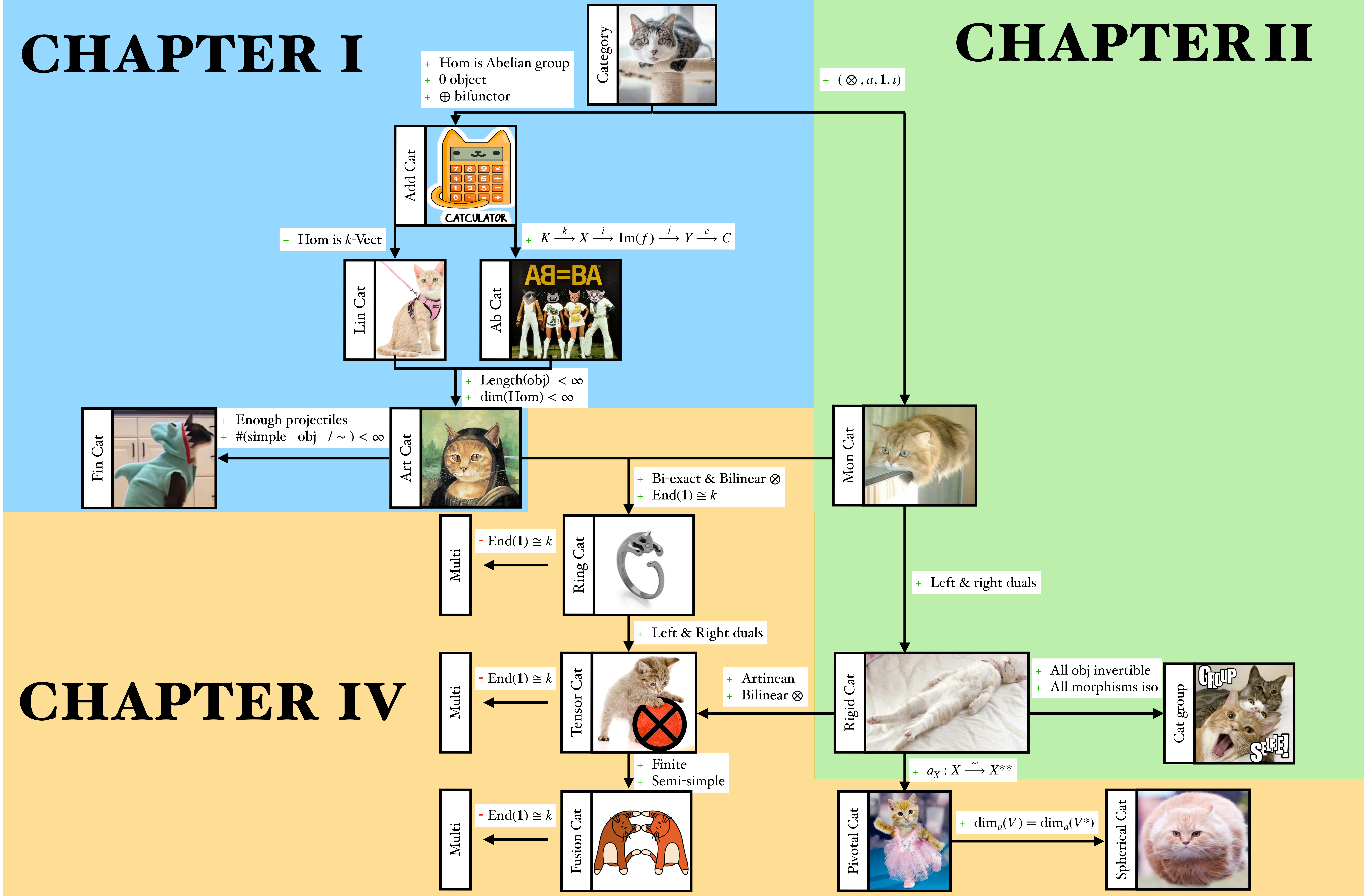
Kleine Seminar

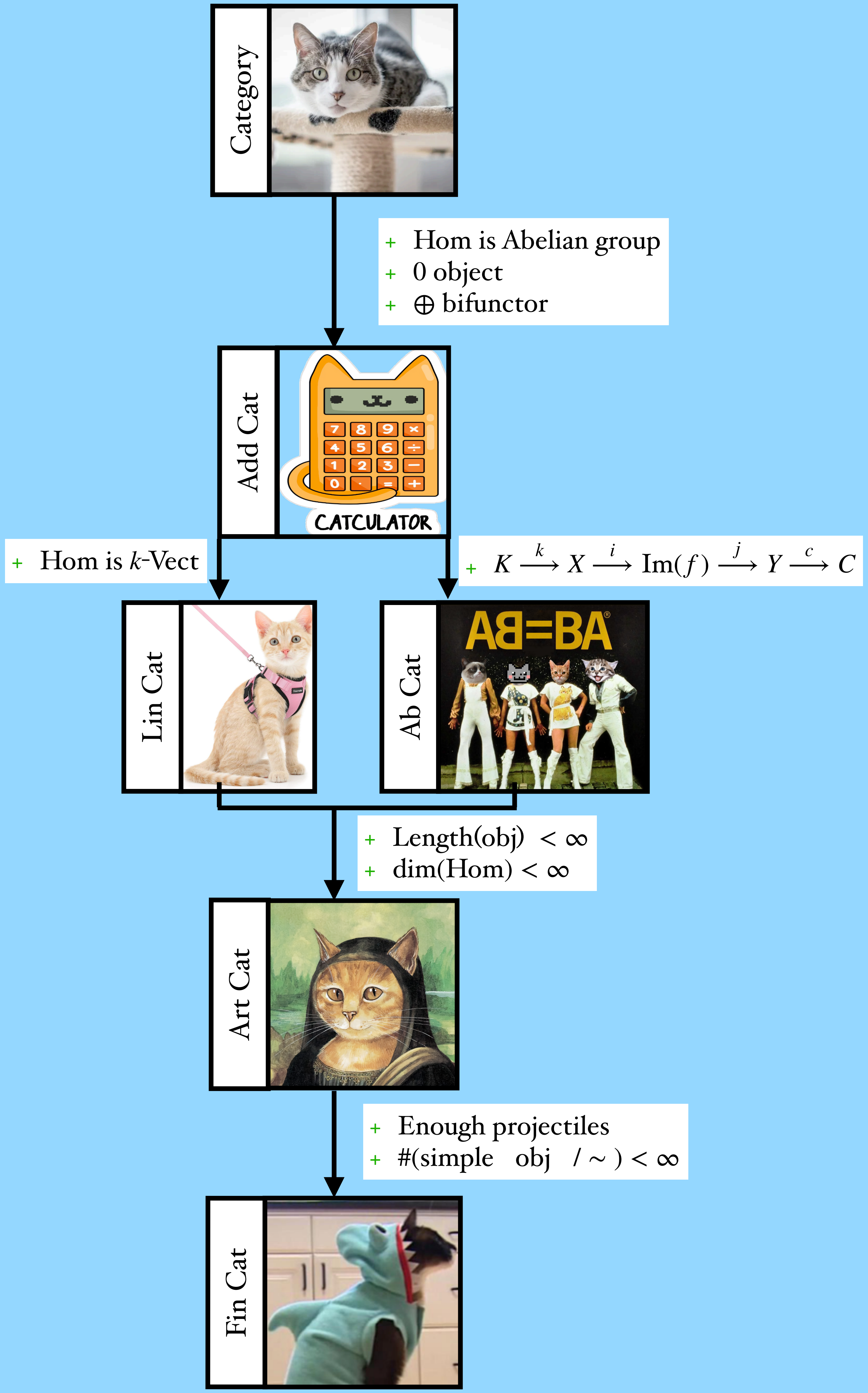
Revision chapters 1 - 7

Gert Vercleyen

Algebras, Coalgebras, Bialgebras, Hopf Algebras


CHAPTER I
CHAPTER II
CHAPTER IV
Category
+ Hom is Abelian group
+ 0 object
+ $\oplus$ bifunctor
+ $(\otimes, a, \mathbf{1}, \iota)$
Add Cat
CATCULATOR
+ Hom is k-Vect
+ $K \xrightarrow{k} X \xrightarrow{i} \mathrm{Im}(f) \xrightarrow{j} Y \xrightarrow{c} C$
Lin Cat
Ab Cat
AB=BA
+ Length(obj) $< \infty$
+ dim(Hom) $< \infty$
Fin Cat
+ Enough projectiles
+ #(simple obj / $\sim$) $< \infty$
Art Cat
Mon Cat
+ Bi-exact & Bilinear $\otimes$
+ $\mathrm{End}(\mathbf{1}) \cong k$
Multi
- $\mathrm{End}(\mathbf{1}) \cong k$
Ring Cat
+ Left & right duals
+ Left & Right duals
Multi
- $\mathrm{End}(\mathbf{1}) \cong k$
Tensor Cat
+ Artinean
+ Bilinear $\otimes$
Rigid Cat
+ All obj invertible
+ All morphisms iso
Cat group
GROUP
SELFIE!
+ Finite
+ Semi-simple
+ $a_X : X \xrightarrow{\sim} X^{**}$
Multi
- $\mathrm{End}(\mathbf{1}) \cong k$
Fusion Cat
Pivotal Cat
+ $\dim_a(V) = \dim_a(V^*)$
Spherical Cat

CHAPTER I ABELIAN CATEGORIES



Def. Kernel & Cokernel

↳ **Def.** Abelian Category

↳ **Def.** Monomorphism & Epimorphism

↳ **Def.** Subobjects & Quotient objects

↳ **Def.** Simple & Semisimple Object/Category

↳ Schur's Lemma

↳ Jordan Holder Theorem & **Def.** Length(Object)

↳ Indecomposable objects & Krull-Schmidt Theorem

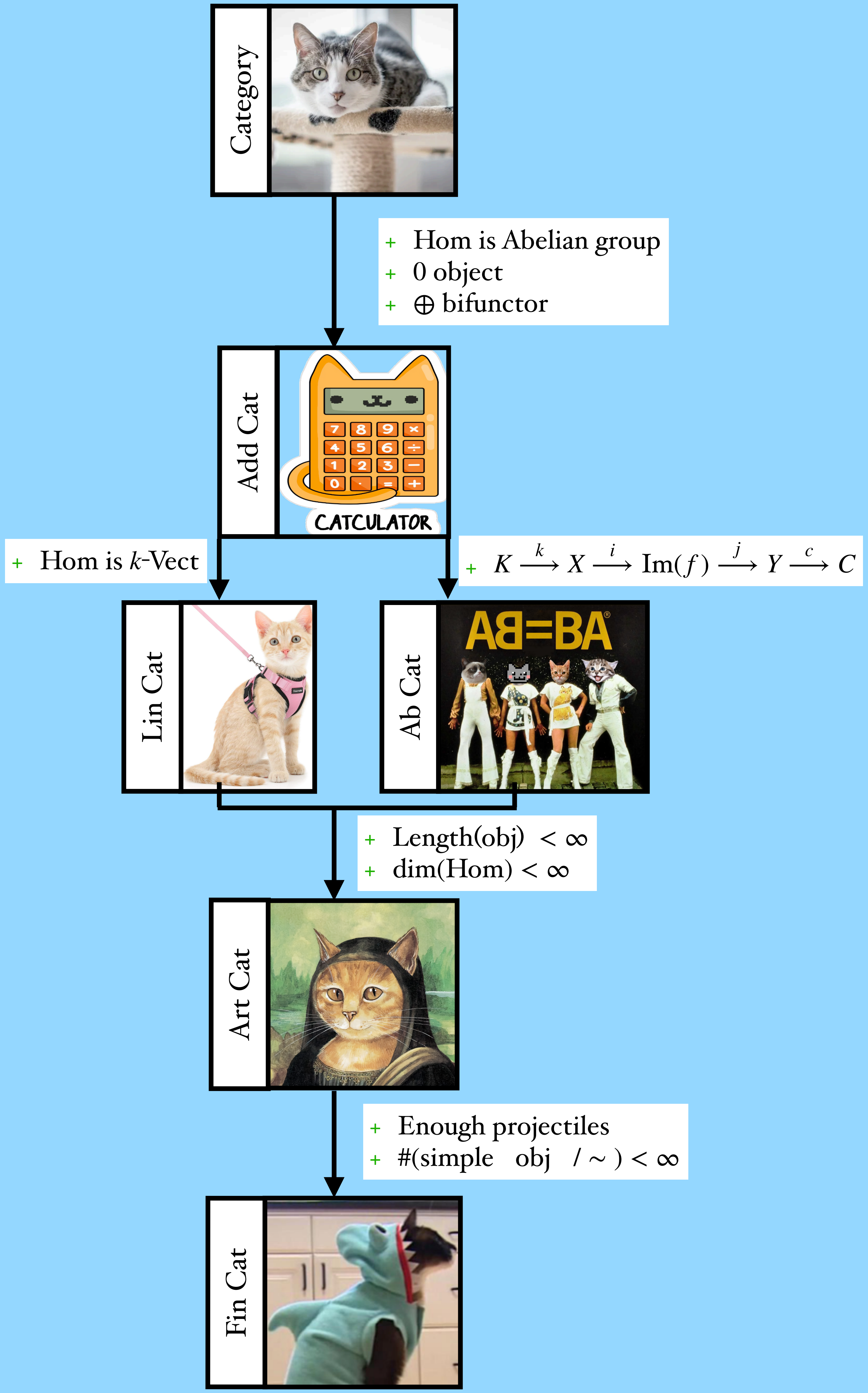
↳ **Def.** (Short) Exact sequences

↳ **Def.** Left/Right exact functors

↳ **Def.** Projective Object/Cover & Injective Object/Hull

CHAPTER I ABELIAN CATEGORIES

First big results



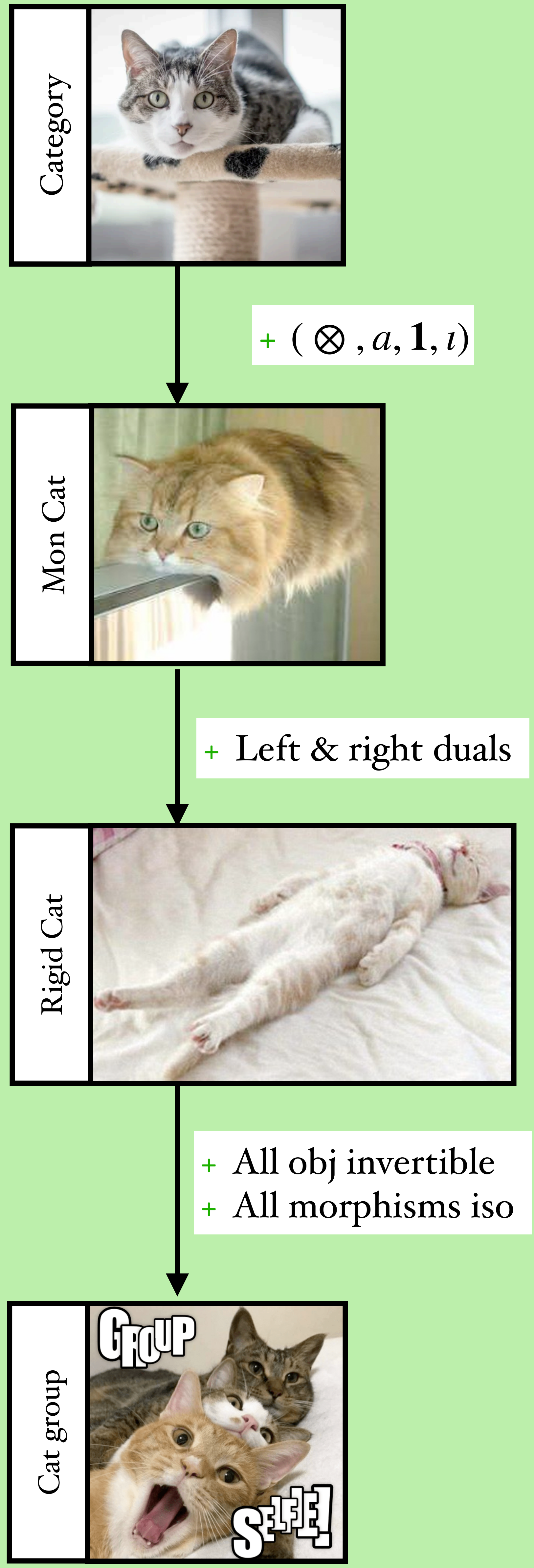
Definition 1.8.5. A $\Bbbk$-linear abelian category $\mathcal{C}$ is said to be *finite* if it is equivalent to the category A–mod of finite dimensional modules over a finite dimensional $\Bbbk$-algebra A.

Corollary 1.8.11. *Let $\mathcal{C}$ be a finite abelian $\Bbbk$-linear category, and let $F : \mathcal{C} \to \mathsf{Vec}$ be an additive $\Bbbk$-linear left exact functor. Then $F = \mathsf{Hom}_{\mathcal{C}}(V, -)$ for some object $V \in \mathcal{C}$.*

Definition 1.9.13. A coalgebra C is *pointed* if any simple right C-comodule is 1-dimensional.

Theorem 1.9.15 (Takeuchi, [**Tak2**]). *Any essentially small locally finite abelian category $\mathcal{C}$ over a field $\Bbbk$ is equivalent to the category C–comod for a unique pointed coalgebra C. In particular, if $\mathcal{C}$ is finite, it is equivalent to the category A–mod for a unique basic algebra A (namely, $A = C^*$).*

CHAPTER II



MONOIDAL CATEGORIES

DEFINITION 2.1.1. A *monoidal category* is a quintuple $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor called the *tensor product* bifunctor, $a : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ is a natural isomorphism:

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \qquad X, Y, Z \in \mathcal{C} \tag{2.1}$$

called the *associativity constraint* (or *associativity isomorphism*), $\mathbf{1} \in \mathcal{C}$ is an object of $\mathcal{C}$, and $\iota : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ is an isomorphism, subject to the following two axioms.

1. The pentagon axiom. The diagram

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "a_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd} \tag{2.2}$$

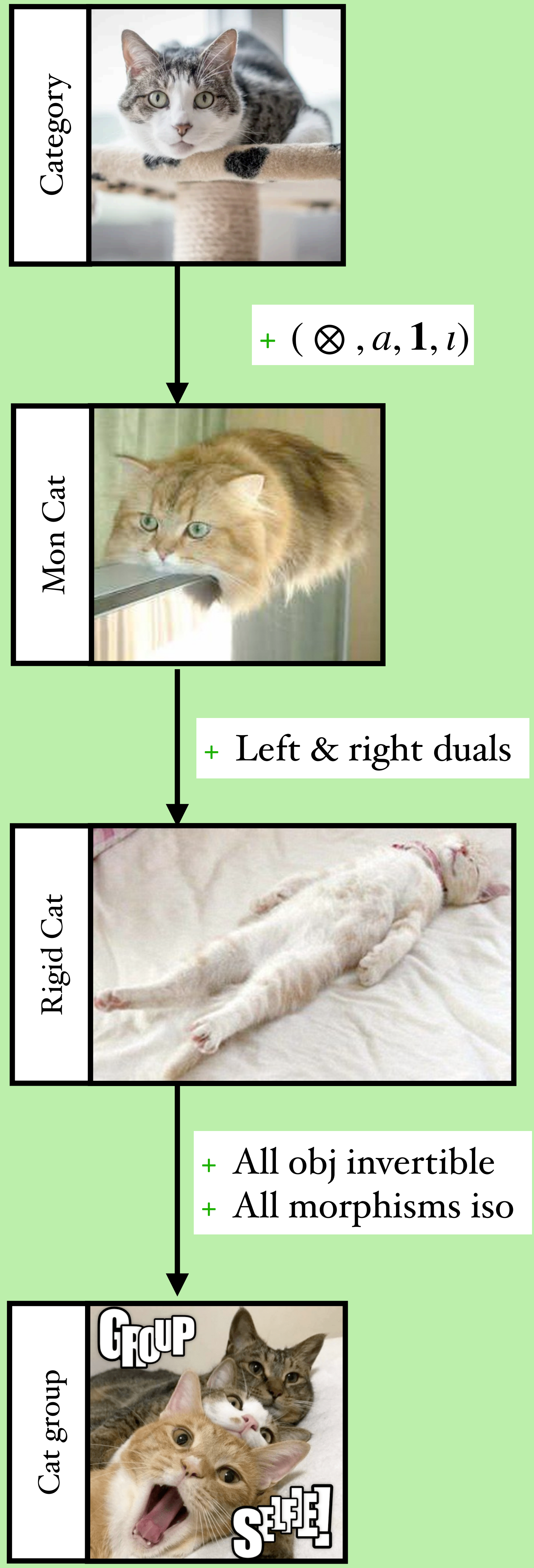
is commutative for all objects W, X, Y, Z in $\mathcal{C}$.

2. The unit axiom. The functors

$$L_{\mathbf{1}} \ : \ X \mapsto \mathbf{1} \otimes X \quad \text{and} \tag{2.3}$$

$$R_{\mathbf{1}} \ : \ X \mapsto X \otimes \mathbf{1} \tag{2.4}$$

CHAPTER II



MONOIDAL CATEGORIES

Note: $U \otimes V \ncong V \otimes U$ in general

Properties of the unit $\mathbf{1}$

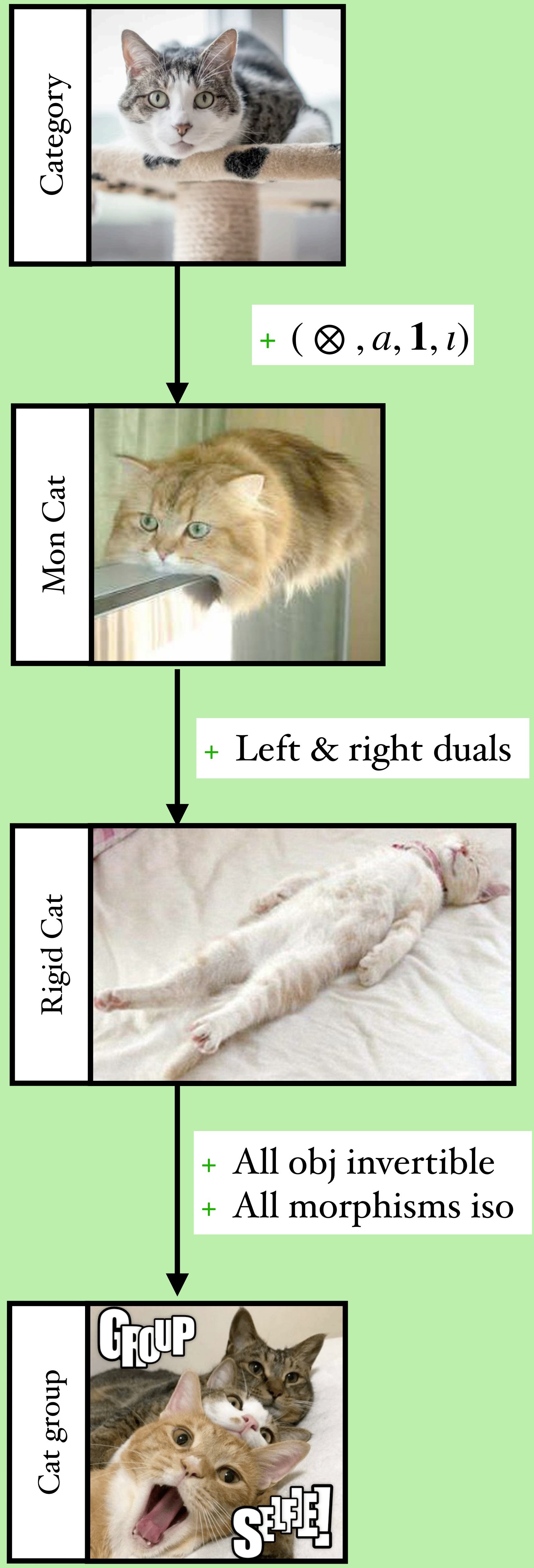
Prop 2.2.6.

$\mathbf{1}$ is unique UTUI

Prop 2.2.10.

End($\mathbf{1}$) is commutative monoid

CHAPTER II



MONOIDAL CATEGORIES

Note on Monoidal Functors

Definition 2.4.1. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ and $(\mathcal{C}^{\wr}, \otimes^{\wr}, \mathbf{1}^{\wr}, a^{\wr}, \iota^{\wr})$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}^{\wr}$ is a pair (F, J), where $F : \mathcal{C} \to \mathcal{C}^{\wr}$ is a functor, and

$$J_{X,Y} : F(X) \otimes^{\wr} F(Y) \xrightarrow{\sim} F(X \otimes Y) \tag{2.22}$$

is a natural isomorphism, such that $F(\mathbf{1})$ is isomorphic to $\mathbf{1}^{\wr}$ and the diagram

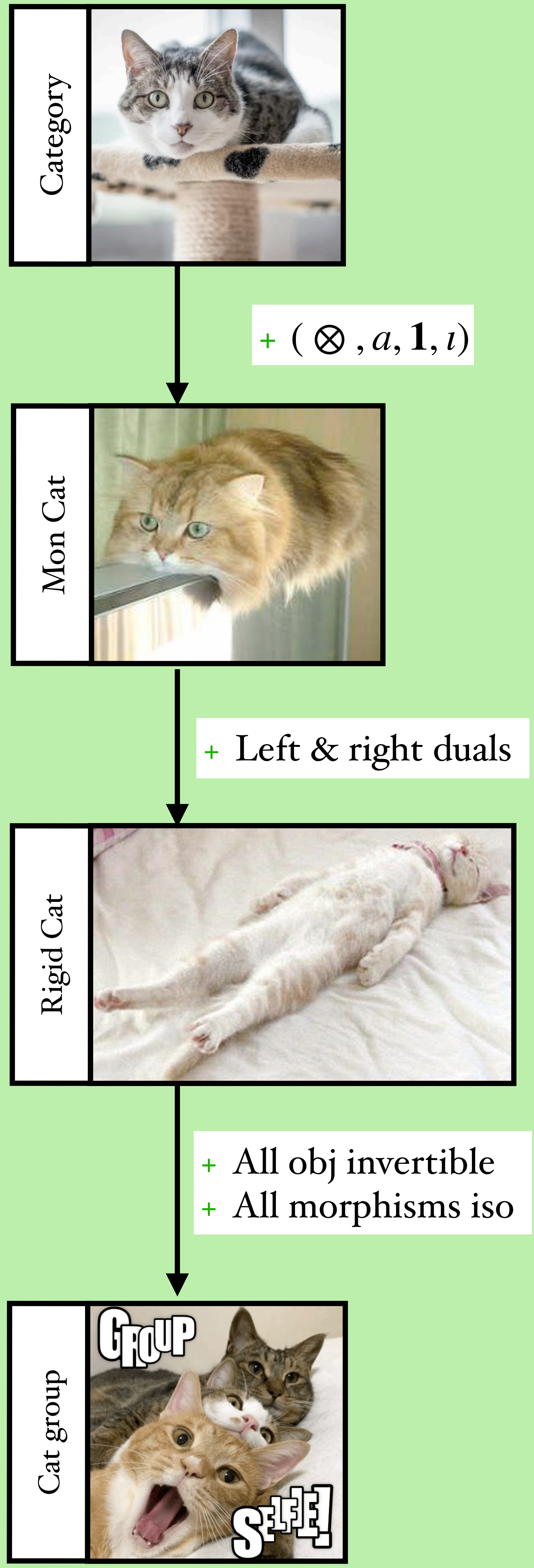
$$\begin{tikzcd}
(F(X) \otimes^{\wr} F(Y)) \otimes^{\wr} F(Z) \arrow[r, "a^{\wr}_{F(X),F(Y),F(Z)}"] \arrow[d, "J_{X,Y} \otimes^{\wr} \mathrm{id}_{F(Z)}"'] & F(X) \otimes^{\wr} (F(Y) \otimes^{\wr} F(Z)) \arrow[d, "\mathrm{id}_{F(X)} \otimes^{\wr} J_{Y,Z}"] \\
F(X \otimes Y) \otimes^{\wr} F(Z) \arrow[d, "J_{X \otimes Y, Z}"'] & F(X) \otimes^{\wr} F(Y \otimes Z) \arrow[d, "J_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(a_{X,Y,Z})"] & F(X \otimes (Y \otimes Z))
\end{tikzcd} \tag{2.23}$$

is commutative for all $X, Y, Z \in \mathcal{C}$ (“the monoidal structure axiom”).

A monoidal functor F is said to be an *equivalence of monoidal categories* if it is an equivalence of ordinary categories.

<u>For given *F* there can be multiple or no *J* at all!</u>

CHAPTER II



MONOIDAL CATEGORIES

2 Important Theorems

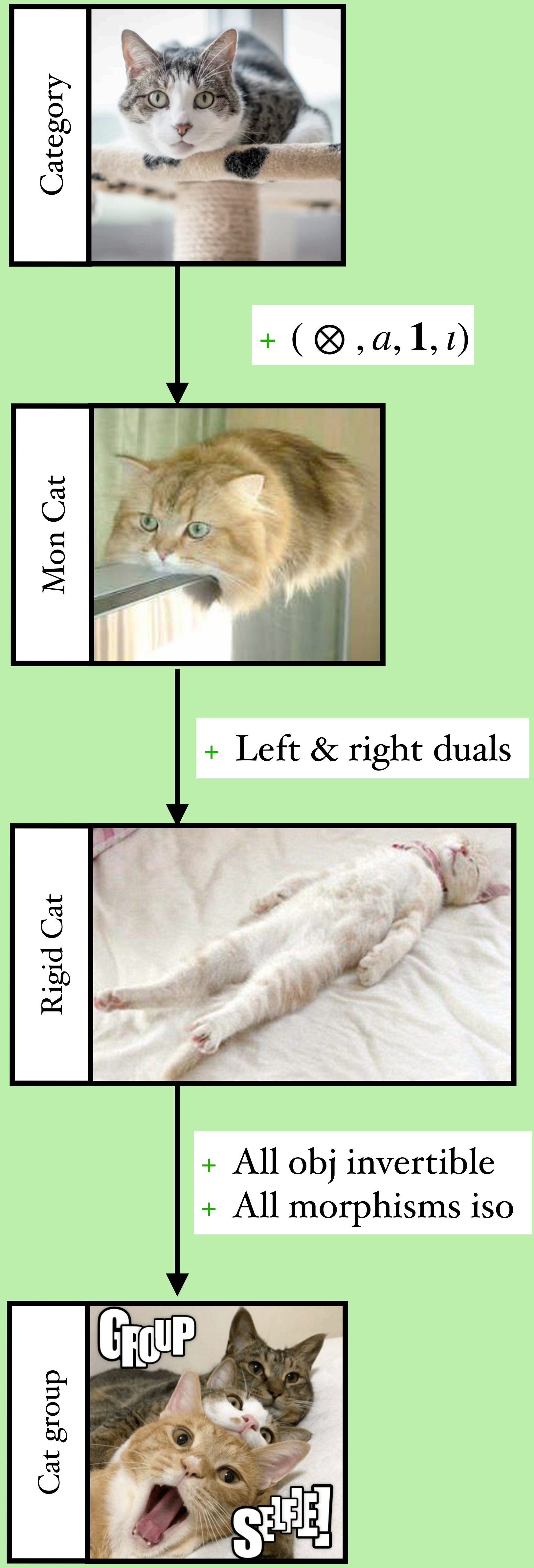
DEFINITION 2.8.1. A monoidal category $\mathcal{C}$ is *strict* if for all objects X, Y, Z in $\mathcal{C}$ one has equalities $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $X \otimes \mathbf{1} = X = \mathbf{1} \otimes X$, and the associativity and unit constraints are the identity maps.

THEOREM 2.8.5. *Any monoidal category is monoidally equivalent to a strict monoidal category.*

Note: equivalent $\neq$ isomorphic!

THEOREM 2.9.2. *(Coherence Theorem) Let $X_1, \ldots, X_n \in \mathcal{C}$. Let P_1, P_2 be any two parenthesized products of $X_1, \ldots, X_n$ (in this order) with arbitrary insertions of the unit object $\mathbf{1}$. Let $f, g : P_1 \to P_2$ be two isomorphisms, obtained by composing associativity and unit isomorphisms and their inverses possibly tensored with identity morphisms. Then $f = g$.*

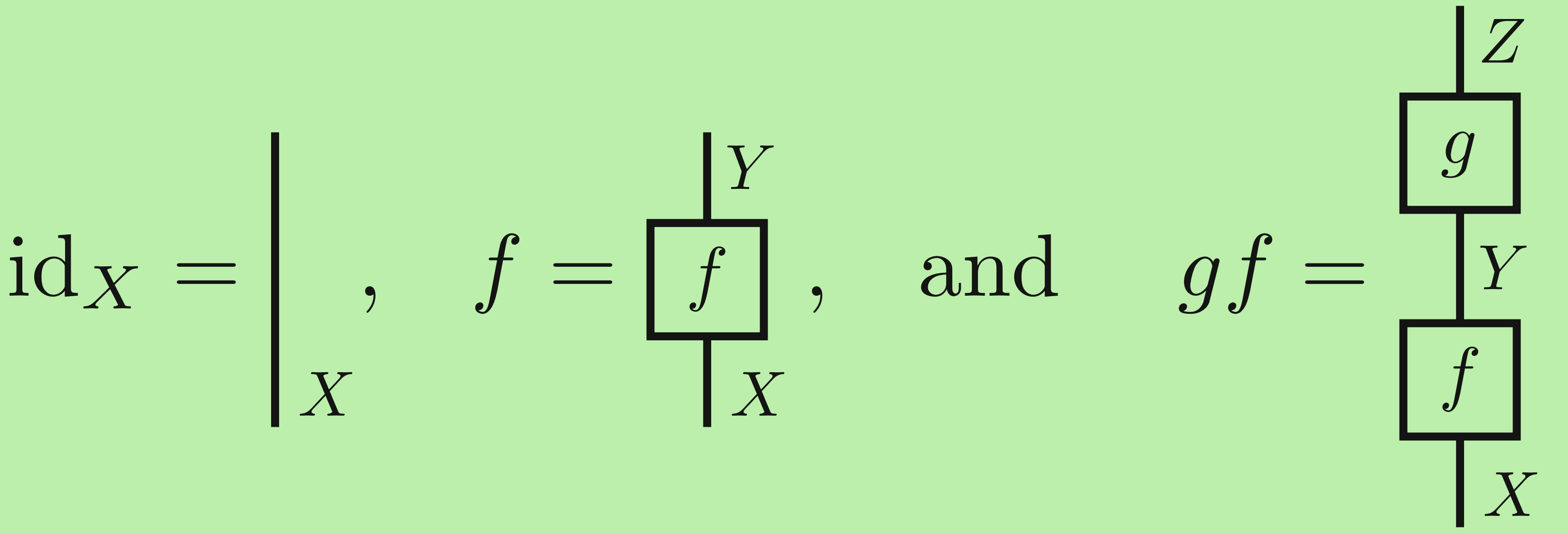
CHAPTER II



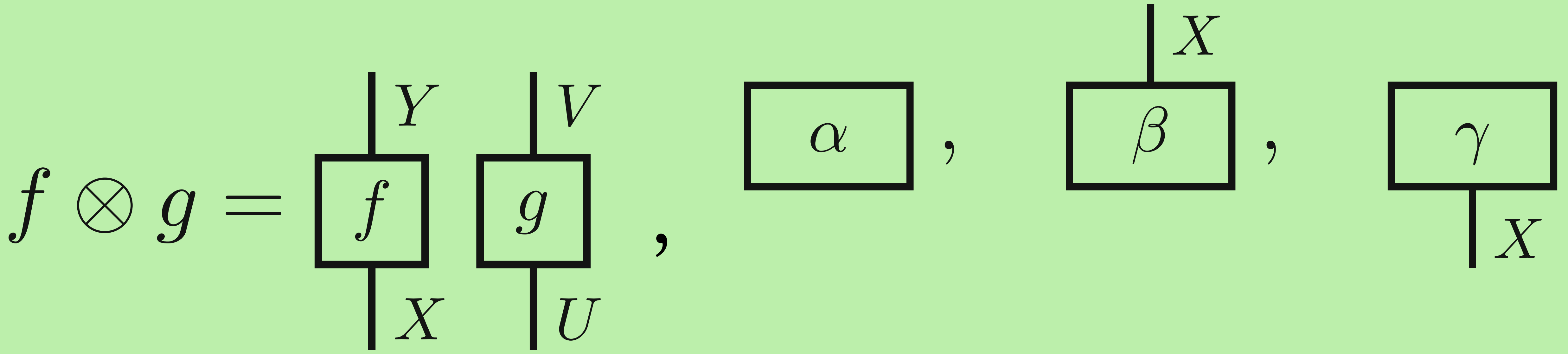
MONOIDAL CATEGORIES

Graphical Language

General Cats:

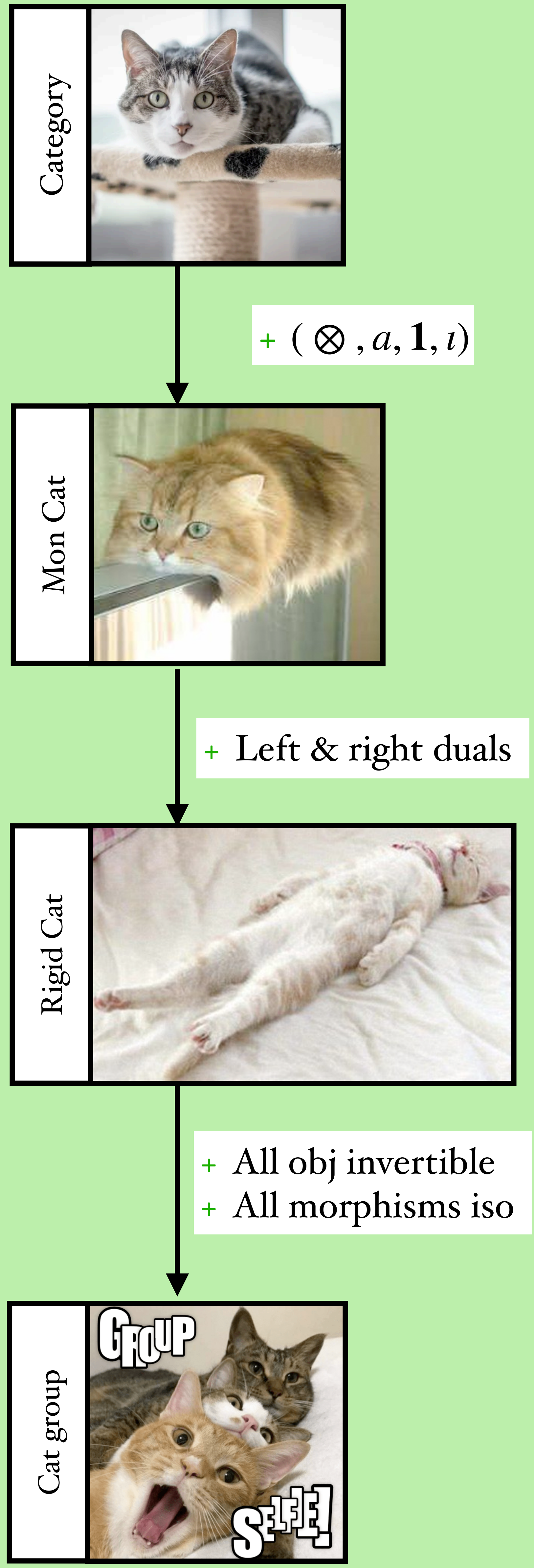


Mon Cats:



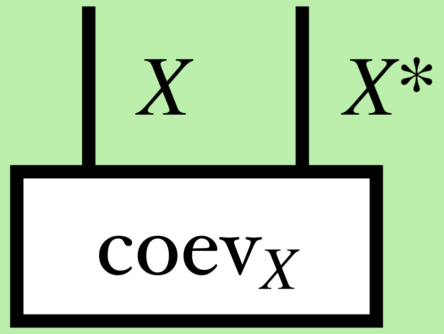
$\mathbf{1} \xrightarrow{\alpha} \mathbf{1}$ $\quad$ $\mathbf{1} \xrightarrow{\beta} X$ $\quad$ $X \xrightarrow{\gamma} \mathbf{1}$

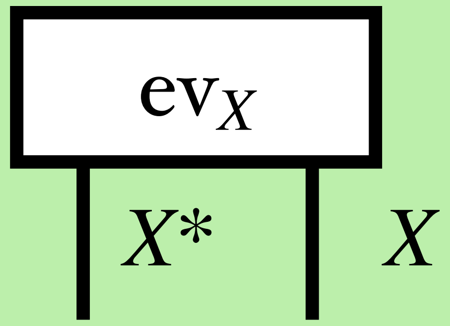
CHAPTER II



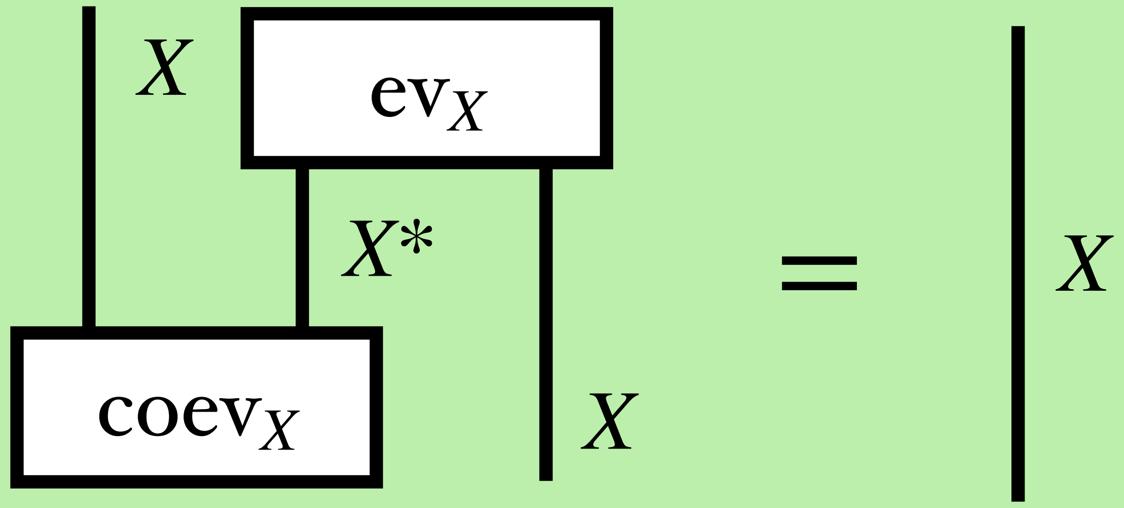
MONOIDAL CATEGORIES

Rigid Cats and Cat Groups

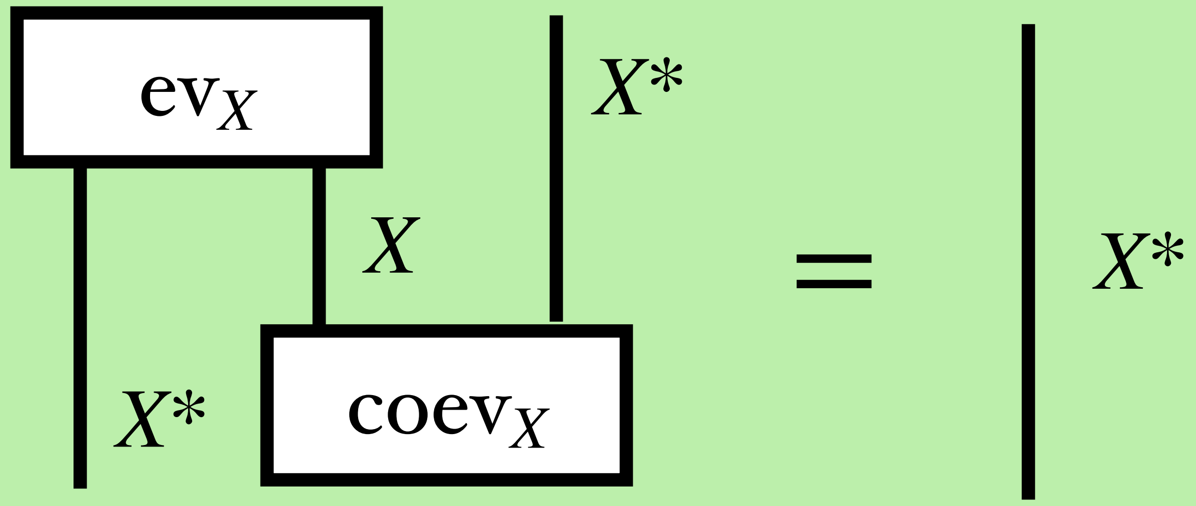
Def. X^* is left dual of X if $\exists$ 

& 

s.t.



&



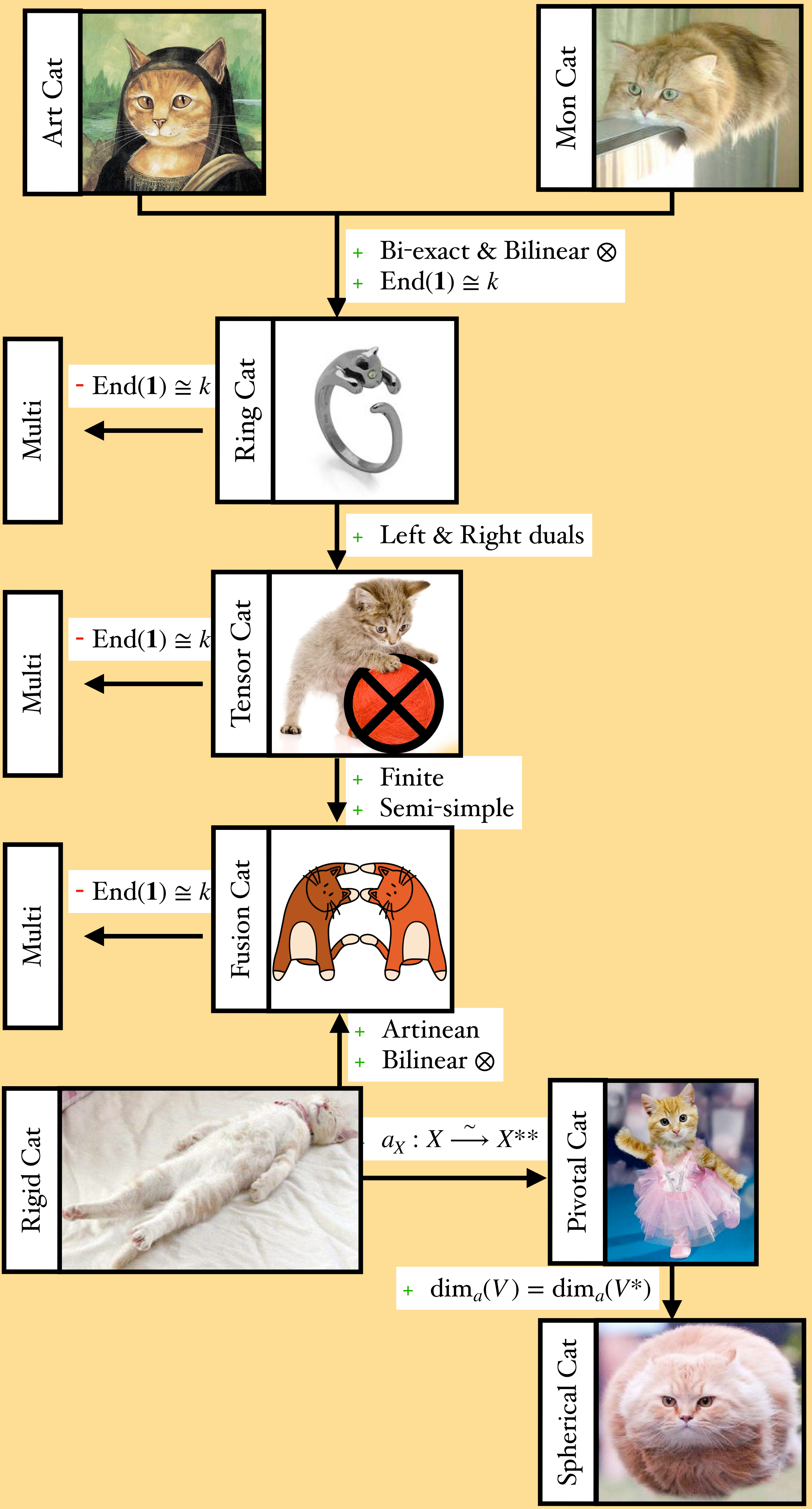
Definition 2.10.11. An object in a monoidal category is called *rigid* if it has left and right duals. A monoidal category $\mathcal{C}$ is called *rigid* if every object of $\mathcal{C}$ is rigid.

CHAPTER III

$\mathbb{Z}_+$ -RINGS

Masterfully explained by world class expert:
I assume its content is still vividly alive in everyones memory

CHAPTER IV



TENSOR CATEGORIES

General results for multi-tensor categories

Thm 4.2.1.

$\otimes$ is biexact

***Thm 4.2.8.**

$$\mathrm{Im}(f_1 \otimes f_2) = \mathrm{Im}(f_1) \otimes \mathrm{Im}(f_2)$$

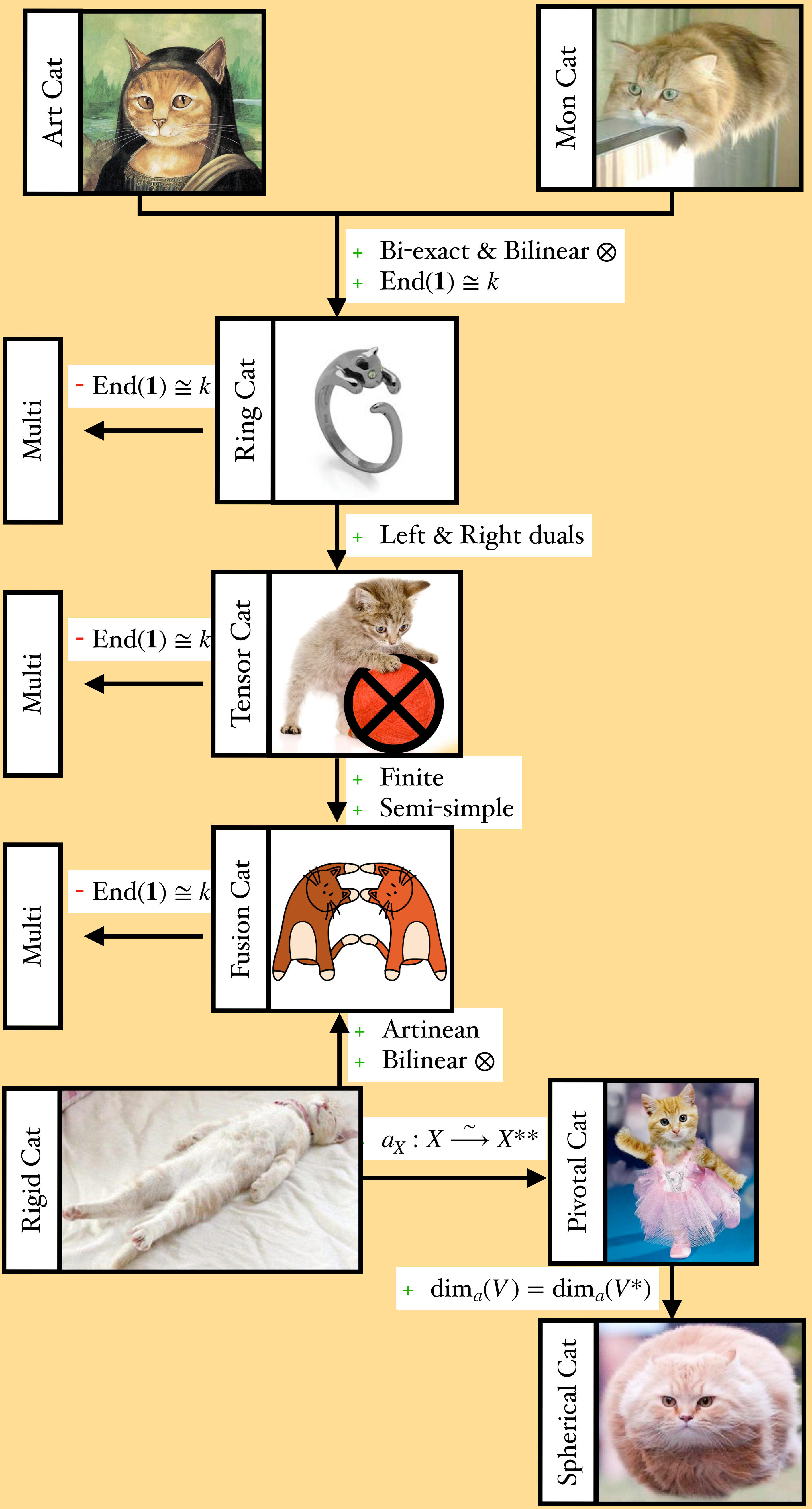
***Thm 4.2.9.**

$^*(-)$ and $(-)^*$ are exact

***Cor 4.3.9.**

coev_X are monos and ev_X are epis

CHAPTER IV



TENSOR CATEGORIES

Properties of the unit 1 of multi-tensor categories

***Cor 4.2.13.**

$\mathbf{1}$ projective $\Leftrightarrow$ C semisimple

***Thm 4.3.1.**

$$\mathrm{End}(\mathbf{1}) \cong k \oplus \ldots \oplus k$$

***Cor 4.3.2.**

$\mathbf{1} \cong \bigoplus_i I_i$ with I_i non-isomorphic indecomposable objects

***Thm 4.3.8.**

C tensor cat $\Rightarrow$ $\mathbf{1}$ simple

***Thm 4.3.8.**

C finite, $\mathrm{Char}(k) = 0$, C $\exists!$ simple object $\mathbf{1}$ $\Rightarrow$ $C \cong \mathsf{Vec}$

CHAPTER V

REPRESENTATION CATEGORIES OF HOPF ALGEBRAS

Reconstruction theory

Def 5.1.1.

A (Quasi-)Fiber functor is a (quasi-)tensor functor to Vec

THEOREM 5.2.3. *The assignments*

$$(\mathcal{C}, F) \mapsto H = \mathsf{End}(F), \quad H \mapsto (\mathsf{Rep}(H), \mathrm{Forget}) \tag{5.1}$$

are mutually inverse bijections between (1) $\boxed{\textit{finite ring categories}}$ *$\mathcal{C}$ with a fiber functor $F : \mathcal{C} \to \mathsf{Vec}$, up to tensor equivalence and isomorphism of tensor functors and (2) isomorphism classes of* $\boxed{\textit{finite dimensional bialgebras}}$ *H over $\Bbbk$.*

THEOREM 5.3.12. *The assignments*

$$(\mathcal{C}, F) \mapsto H = \mathsf{End}(F), \quad H \mapsto (\mathsf{Rep}(H), \mathrm{Forget}) \tag{5.5}$$

are mutually inverse bijections between (1) equivalence classes of $\boxed{\textit{finite tensor categories}}$ *$\mathcal{C}$ with a fiber functor F, up to tensor equivalence and isomorphism of tensor functors, and (2) isomorphism classes of* $\boxed{\textit{finite dimensional Hopf algebras}}$ *over $\Bbbk$.*

CHAPTER V

REPRESENTATION CATEGORIES OF HOPF ALGEBRAS

Reconstruction theory in the infinite setting

Theorem 5.4.1. *The assignments*

$$(\mathcal{C}, F) \mapsto H = \mathsf{Coend}(F), \quad H \mapsto (H - \mathrm{Comod}, \mathrm{Forget}) \tag{5.9}$$

are mutually inverse bijections between the following pairs of sets:

(1) *ring categories* $\mathcal{C}$ *over* $\Bbbk$ *with a fiber functor* F*, up to tensor equivalence and isomorphism of tensor functors, and bialgebras over* $\Bbbk$*, up to isomorphism;*

(2) *ring categories* $\mathcal{C}$ *over* $\Bbbk$ *with left duals with a fiber functor* F*, up to tensor equivalence and isomorphism of tensor functors, and bialgebras over* $\Bbbk$ *with an antipode, up to isomorphism;*

(3) *tensor categories* $\mathcal{C}$ *over* $\Bbbk$ *with a fiber functor* F*, up to tensor equivalence and isomorphism of tensor functors, and Hopf algebras over* $\Bbbk$*, up to isomorphism.*

CHAPTER VI

FINITE TENSOR CATEGORIES

Self-study

CHAPTER VII

MODULE CATEGORIES

DEFINITION 7.1.1. A *left module category* over $\mathcal{C}$ is a category $\mathcal{M}$ equipped with an *action (or module product) bifunctor* $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and a natural isomorphism

$$m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M), \qquad X, Y \in \mathcal{C},\ M \in \mathcal{M}, \tag{7.1}$$

called *module associativity constraint* such that the functor $M \mapsto \mathbf{1} \otimes M : \mathcal{M} \to \mathcal{M}$ is an autoequivalence, and the *pentagon diagram*:

$$\begin{tikzcd}[column sep=small]
& ((X \otimes Y) \otimes Z) \otimes M \arrow[dl, "a_{X,Y,Z} \otimes \mathsf{id}_M"'] \arrow[dr, "m_{X \otimes Y, Z, M}"] & \\
(X \otimes (Y \otimes Z)) \otimes M \arrow[d, "m_{X, Y \otimes Z, M}"'] & & (X \otimes Y) \otimes (Z \otimes M) \arrow[d, "m_{X, Y, Z \otimes M}"] \\
X \otimes ((Y \otimes Z) \otimes M) \arrow[rr, "\mathsf{id}_X \otimes m_{Y,Z,M}"] & & X \otimes (Y \otimes (Z \otimes M))
\end{tikzcd} \tag{7.2}$$

is commutative for all objects X, Y, Z in $\mathcal{C}$ and M in $\mathcal{M}$.

PROPOSITION 7.1.3. *There is a bijective correspondence between structures of a $\mathcal{C}$-module category on $\mathcal{M}$ and monoidal functors $F : \mathcal{C} \to \mathsf{End}(\mathcal{M})$.*

CHAPTER VII MODULE CATEGORIES

Let $\mathcal{C}$ be a multitensor category over $\Bbbk$.

DEFINITION 7.3.1. A *module category* over $\mathcal{C}$ (or $\mathcal{C}$-module category) is a locally finite abelian category $\mathcal{M}$ over $\Bbbk$ which is equipped with a structure of a $\mathcal{C}$-module category, such that the module product bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is bilinear on morphisms and exact in the first variable.

Let $\mathsf{End}_l(\mathcal{M})$ be the category of left exact functors from $\mathcal{M}$ to $\mathcal{M}$.

PROPOSITION 7.3.3. *There is a bijection between structures of a $\mathcal{C}$-module category on $\mathcal{M}$ and tensor functors $F : \mathcal{C} \to \mathsf{End}_l(\mathcal{M})$.*[1]

CHAPTER VII

MODULE CATEGORIES

THEOREM. A graduate student dividing time between productive tasks and browsing cat pictures will spend too much time doing the latter.

COROLLARY I didn't have enough time to include the rest of chapter 7.

- Pictures of graphical calculus were copied from
  Turaev and Virelizier: Monoidal Categories and Topological Field Theory,
  which itself was copied from an unspecified source on the internet