Tensor Categories, Chapter 7

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Outline

Highlights of last time: (7.8) – (7.10)

Fix $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ a tensor category. We wish to "understand" left $\mathcal{C}$-module categories $\mathcal{M} = (\mathcal{M}, \otimes, m, l)$.

- Algebras in $\mathcal{C}$: $A = (A, m, u)$
- Right A-modules: $M = (M, p)$
- The left $\mathcal{C}$-module category $\mathrm{Mod}_{\mathcal{C}}(A)$
- Notion of Morita equivalence of algebras
- Internal Hom in a module category $\mathcal{M}$: $\underline{\mathrm{Hom}}(M_1, M_2) \in \mathcal{C}$, $M_i \in \mathcal{M}$

$$\underbrace{\mathrm{Hom}_{\mathcal{M}}(X \otimes M_1, M_2)}_{}\ \overset{-\otimes M_1 \qquad\qquad \underline{H}(M_1,-)}{\cong}\ \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M_1, M_2)) \qquad (🐱)$$

- Products: $\underline{\mathrm{Hom}}(M_2, M_3) \otimes \underline{\mathrm{Hom}}(M_1, M_2) \to \underline{\mathrm{Hom}}(M_1, M_3)$
- Algebras $A_M := \underline{\mathrm{Hom}}(M, M)$, right-mods $\underline{\mathrm{Hom}}(M, N)$, $(M, N \in \mathcal{M})$

(Cat) $X = \mathbb{1}$, $M_i = M$ $\qquad$ $H_{\mathcal{M}}(\mathbb{1} \otimes M, M) \simeq \mathrm{Hom}(\mathbb{1}, A_M)$

$$\mathrm{id} \in H_{\mathcal{M}}(\mathbb{1}\otimes M, M) \longmapsto u_M$$

Characterization of module categories in terms of algebras

Assume $\mathcal{C}$ finite, $\mathcal{M}$ a left $\mathcal{C}$-module category with $M \in \mathcal{M}$ satisfying

(🐈‍⬛) $\underline{\mathrm{Hom}}(M, -)$ is right exact

(🐈) for all $N \in \mathcal{M}$ exists $X \in \mathcal{C}$ and a surjection $X \otimes M \to N$.

Theorem (7.10.1)

The functor $F_M : \mathcal{M} \to \mathrm{Mod}_{\mathcal{C}}(A_M)$ given by

$$F_M(N) = \underline{\mathrm{Hom}}(M, N)$$

is an equivalence of cats.

$$A_M := \underline{\mathrm{Hom}}(M, M)$$

Goals of today

- Discuss the proof.
- Discuss situations in which such M exists.
- Category of module functors (7.11).
- Dual tensor categories (7.12).
- Categorical Morita equivalence (7.12).

Outline

8: Alexis / 2 more

14/02	Revision :	Gert
21/02	8.1 — 8.5 :	Alexis
28/02	8.6 — 8.9 :	Michiel
07/03	8.10 — 8.14 :	Sam?

9:

14/03	9.1 – 9.5 :	Jari
21/03	9.6 – 9.9 :	Marcelo
28/03	9.10 9.12 :	Sigi
04/04	Fusion Cats :	Gert

Proof of 7.10.1: $A = A_M$, $F = F_M$ $\underline{H}(M, X\otimes N) \simeq X \otimes \underline{H}(M,N)$

(1) $\mathrm{Hom}_{\mathcal{M}}(N_1, N_2) \cong \mathrm{Hom}_A(F(N_1), F(N_2))$ for $\underline{N_1 = X \otimes M}$, any N_2.

$$FN_1 = \underline{H}(M, X\otimes M) \simeq X \otimes \underline{H}(M,M) = X \otimes A \qquad \forall X.$$

$$\therefore H_A(FN_1, FN_2) \simeq H_A(X\otimes A, FN_2) \simeq H_e(X, FN_2) = H_e(X, \underline{H}(M, N_2))$$

$$\simeq H_{\mathcal{M}}(\underbrace{X\otimes M}_{N_1}, N_2)$$

adj Forget.

(2) $\mathrm{Hom}_{\mathcal{M}}(N_1, N_2) \cong \mathrm{Hom}_A(F(N_1), F(N_2))$ for all N_1, N_2.

(Ginger) $Y\otimes M \to X \otimes M \to N_1 \to 0$

$$\therefore \begin{array}{ccccccc} H_{\mathcal{M}}(Y\otimes M, N_2) & \leftarrow & H_{\mathcal{M}}(X\otimes M, N_2) & \leftarrow & H_{\mathcal{M}}(N_1, N_2) & \leftarrow & 0 \\ \downarrow \wr & & \downarrow \wr & & \downarrow F & & \\ H_A(Y\otimes A, FN_2) & \leftarrow & H_A(X\otimes A, FN_2) & \leftarrow & H_A(FN_1, FN_2) & \leftarrow & 0 \end{array}$$

(3) The functor F is essentially surjective.

$L \in \mathrm{Mod}_e(A)$, L is coker of $Y\otimes A \xrightarrow{f} X\otimes A \to L \to 0$

$$\underline{H_A(Y\otimes A, X\otimes A)} \simeq H_A(F(Y\otimes M), F(X\otimes M)) \simeq H_{\mathcal{M}}\underline{(Y\otimes M, X\otimes M)}$$

$$Y\otimes M \xrightarrow{\tilde{f}} X\otimes M \to N \to 0$$

$L = F(N)$, N coker $(\tilde{f})$. aplying F

Condition (🐈) on M of (7.10.1)

$\underline{H}_j := \underline{\mathrm{Hom}}(M, N_j)$

Proposition (7.10.3)

[(7.6.9) $\underline{H}(\underline{M}, -)$ is a mod funct.

If (a) M projective or (b) $\mathcal{M}$ exact then (🐈) is satisfied.

Note that (cat) eq^n $\underline{H}(M, -)$ is right adj to $- \otimes M$ $\Rightarrow$ $\underline{H}(M, -)$ is left exact.

"Small Yoneda $\mathcal{C}$ abelia, $A, B, C \in \mathcal{C}$

if $H_{\mathcal{C}}(X, A) \xrightarrow{f \circ -} H_{\mathcal{C}}(X, B) \xrightarrow{g \circ -} H_{\mathcal{C}}(x, c)$ exact $\Rightarrow$ $A \xrightarrow{f} B \xrightarrow{g} C$ exact."

$0 \to N_1 \to N_2 \to N_3 \to 0$ s.e.s. in $\mathcal{M}$, if $X \otimes M$ proj. $\forall X \in \mathcal{C}$

$$\Rightarrow \quad 0 \to H_{\mathcal{M}}(X \otimes M, N_1) \to H_{\mathcal{M}}(X \otimes M, N_2) \to H_{\mathcal{M}}(X \otimes M, N_3) \to 0$$

$$\quad\quad\ \ \wr\!\wr \qquad\qquad\qquad\qquad \wr\!\wr \qquad\qquad\qquad\qquad \wr\!\wr$$

$$0 \to H_{\mathcal{C}}(X, \underline{H}_1) \to H_{\mathcal{C}}(X, \underline{H}_2) \to H_{\mathcal{C}}(X, \underline{H}_3) \to 0$$

$\Rightarrow$ $0 \to \underline{H}_1 \to \underline{H}_2 \to \underline{H}_3 \to 0$ exact!

Condition (🐈) on M of (7.10.1)

(7.6.9)

Proposition (7.10.4)

If either (a) or (b) holds and $\mathcal{M}$ indecomposable, then (🐈) is equiv to $[M]$ generates $\mathrm{Gr}(\mathcal{M})$ as a $\mathbb{Z}_+$-mod over $\mathrm{Gr}(\mathcal{C})$.

Section 7.7!

Corollary (7.10.5)

(i) Let $\mathcal{M}$ be finite. Then $\mathcal{M} \cong \mathrm{Mod}_{\mathcal{C}}(A)$ for some $A \in \mathcal{C}$.

(ii) Let $\mathcal{M}$ exact and $M \in \mathcal{M}$ s.t. $[M]$ generates $\mathrm{Gr}(\mathcal{M})$ as a $\mathbb{Z}_+$-module over $\mathrm{Gr}(\mathcal{C})$. Then $\mathcal{M} \cong \mathrm{Mod}_{\mathcal{C}}(A)$ where $A = \underline{\mathrm{Hom}}(M, M)$.

(7.6.9) F mod. functor, $\mathcal{M}$ exact $\Rightarrow$ F exact

$F : \mathcal{M} \to \mathcal{N}$; $P \otimes M = 0 \Rightarrow M = 0$

Outline

Section 7.11

$\mathcal{F}un(\mathcal{M}, \mathcal{M})$!

Definition

Let $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ denote the full subcat of the cat of module functors consiting of right exact functors.

Proposition

The following is true regarding $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$:

(7.11.1) If $\mathcal{M}_j = \mathrm{Mod}_{\mathcal{C}}(A_j)$, then $\mathrm{Bimod}_{\mathcal{C}}(A_1, A_2) \cong \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$.

(7.11.3) If $\mathcal{M}_j, j = 1, 2, 3$ are exact the composition below is bi-exact:

(7.6.9)! $$\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_2, \mathcal{M}_3) \times \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2) \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_3).$$

(7.11.5) Any $F \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ maps proj into proj if $\mathcal{M}_j$ are exact.

(7.11.6) If $\mathcal{C}$ is finite and $\mathcal{M}_j$ exact then $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is finite.

Sketch of proof of Proposition

An (A_1, A_2)-bimod is a triple (M, p, q), $M \in \mathcal{C}$

s.t. 1) (M, p) left A_1-mod.

2) (M, q) right A_2-mod

3)
$$\begin{tikzcd}
(A_1 \otimes M) \otimes A_2 \arrow[d, "p \otimes 1"'] \arrow[rr, "\simeq"] & & A_1 \otimes (M \otimes A_2) \arrow[d, "1 \otimes q"] \\
M \otimes A_2 \arrow[r, "q"] & M & A_1 \otimes M \arrow[l, "p"']
\end{tikzcd}$$

the equiv. is realized by

$$\mathrm{Bimod}(A_1, A_2) \ni M \longmapsto - \otimes_{A_1} M$$

Outline

Dual tensor categories

Proposition .

If $F : \mathcal{M} \to \mathcal{N}$ a module functor, $\mathcal{M}, \mathcal{N}$ exact, then G, the right adjoint of F is a module functor.

Definition

The cat $\mathcal{C}^*_{\mathcal{M}} := \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is the dual tensor cat to $\mathcal{C}$ w.r.t. $\mathcal{M}$.

For nice properties: $\mathcal{M}$ exact!

Dual tensor categories

Theorem (7.12.11)

Suppose $\mathcal{M}$ is faithful. Then $\mathcal{C}$ is equivalent to $(\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}}$.

$$\mathcal{M} \text{ exact} \rightsquigarrow \mathcal{M} = \bigoplus_{i \in I} \mathcal{M}_i \quad \text{indecomp.}$$

$$\mathcal{C} \ni \mathbb{1} = \bigoplus_{i \in I} \mathbb{1}_i$$

$\mathcal{M}$ faithful if each $\mathbb{1}_i$ acts by a non-zero functor on $\mathcal{M}$.

$$M \longmapsto \mathbb{1}_i \otimes M \in \mathcal{M}$$

Categorical Morita

Definition

Let $\mathcal{C}, \mathcal{D}$ be tensor categories. They are categorically Morita equivalent if there exists an exact module cat $\mathcal{M}$ with $\mathcal{D}^{\mathrm{op}} \cong \mathcal{C}^*_{\mathcal{M}}$.

Proposition

Categorical Morita is an equivalence relation.