Tensor Categories, Chapter 7

Marcelo De Martino

Ghent University

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Outline

Previously ...

§7.1

- Definition of (left) module category over $\mathcal{C}$: $\mathcal{M} = (\mathcal{M}, \otimes, m, l)$
- Structures on $\mathcal{M}$ vs. monoidal functors $F : \mathcal{C} \to \operatorname{End}(\mathcal{M})$
- Abstract nonsense: $\operatorname{Hom}_{\mathcal{M}}(X^* \otimes M, N) \cong \operatorname{Hom}_{\mathcal{M}}(M, X \otimes N)$

§7.2

- $\mathcal{C}$-module functors: $(F, s) \in \operatorname{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$
- Category of $\mathcal{C}$-module functors

§7.3

- Module cat's for $\mathcal{C}$ (multi)tensor

§7.5

- Exact module categories

... previously, still.

§7.6: Assume $\mathcal{M}$ is an exact $\mathcal{C}$-module category

- $\mathcal{M}$ has enough projectives
- $\mathcal{M}$ is completely reducible: $\mathcal{M} = \oplus_{i\in I}\mathcal{M}_i$, $\mathcal{M}_i$ indecomposable cat's
- $F : \mathcal{M}_1 \to \mathcal{M}_2$ additive module functor, $\mathcal{M}_1$ exact $\Rightarrow$ F exact

Proposition assume $\mathrm{End}(\mathbb{1}) = k$

If $P \in \mathcal{C}$ nonzero, $X \in \mathcal{M}$ and $P \otimes X = 0$ then $X = 0$

(4.3.9)
$$\begin{tikzcd} \mathbb{1} \arrow[r, "coev'"] \arrow[rr, bend right] & {}^*P \otimes P \arrow[r] & \mathbb{1} \end{tikzcd} \quad \in k^\times$$

$$X \cong \mathbb{1}\cdot X \xrightarrow{\text{mono.}} ({}^*P \otimes P)\cdot X \xrightarrow{\sim} {}^*P\cdot(P\cdot X) = 0$$

<u>Warning</u>: $\mathrm{Vec} \times \mathrm{Vec}$: $(U,0)\cdot(0,W) = (0,0)$

Outline

Section 7.7

Proposition

$\mathcal{M}/\mathcal{C}$ indecomposable exact module category. Then, $\mathrm{Gr}(\mathcal{M})$ is an irreducible $\mathbb{Z}_+$-module over $\mathrm{Gr}(\mathcal{C})$.

(Omit !!)

Outline

Algebras in $\mathcal{C}$

Assume $\mathcal{C}$ is a (multi)tensor category. We will describe a general construction to obtain module categories over $\mathcal{C}$.

Definition

A triple (A, m, u) with $A \in \mathcal{C}$, $m : A \otimes A \to A$, $u : \mathbb{1} \to A$ is called an *algebra in $\mathcal{C}$* if the following diagrams commute:

$$\begin{tikzcd}
(A \cdot A) \cdot A \arrow[rr, "a"] \arrow[d, "m \cdot"'] & & A \cdot (A \cdot A) \arrow[d, "\cdot m"] \\
A \cdot A \arrow[dr] & \# & A \cdot A \arrow[dl] \\
& A &
\end{tikzcd}$$

(assoc.)

$$\begin{tikzcd}
1 \cdot A \arrow[r, "\ell_A"] \arrow[d, "u \cdot"'] \arrow[dr, phantom, "\#"] & A \arrow[d, equal, "\mathrm{id}_A"] \\
A \cdot A \arrow[r, "m"] & A
\end{tikzcd}$$

(unit)

$$\begin{tikzcd}
A \cdot 1 \arrow[r] \arrow[d, "\cdot u"'] \arrow[dr, phantom, "\#"] & A \arrow[d, equal] \\
A \cdot A \arrow[r] & A
\end{tikzcd}$$

Examples of Algebras

- Fun(G), functions on a finite group G is an algebra in Rep(G).

$$m(f \otimes g) = fg \quad , \quad u : \mathrm{triv} \ni \lambda \longmapsto u_\lambda : g \mapsto \lambda \ \forall g.$$

Proposition

If $X \in \mathcal{C}$ then $A = X \otimes X^*$ is an algebra in $\mathcal{C}$.

$$u : \quad \mathbb{1} \xrightarrow{\mathrm{coev}_X} A = X \cdot X^*$$

$$m : \quad A \cdot A \simeq X \cdot (X^* \cdot X) \cdot X^* \xrightarrow{\cdot\, \mathrm{ev}_X \,\cdot} (X \cdot \mathbb{1}) \cdot X^* \simeq A$$

Modules over algebras in $\mathcal{C}$

Definition

A *right module* over (A, m, u) in $\mathcal{C}$ is a pair (M, p) with $M \in \mathcal{C}$ and $p : M \otimes A \to M$ such that the following diagrams commute:

$$
\begin{tikzcd}
(M \cdot A) A \arrow[rr, "a"] \arrow[d, "p \cdot"'] & & M \cdot (A \cdot A) \arrow[d, "\cdot m"] \\
M \cdot A \arrow[dr, "p"] & & M \cdot A \arrow[dl, "p"'] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \cdot \mathbb{1} \arrow[r] \arrow[d, "\cdot u"'] & M \arrow[d, equal] \\
M \cdot u \arrow[r, "p"] & M
\end{tikzcd}
$$

Proposition

If (M, p) a right $\mathcal{C}$-module then $({}^*M, q)$ is a left $\mathcal{C}$-module with q the image of p under:

$$\varphi : \mathrm{H}_e(M \cdot A, M) \cong \mathrm{H}_e({}^*M, {}^*A \cdot {}^*M) \cong \mathrm{H}_e(A \cdot {}^*M, {}^*M)$$

$$q = \varphi(p)$$

Properties of Algebras

need $(\mathcal{M}, \tilde{\otimes}, \tilde{m}, \tilde{\ell})$ + diagr + abelian, rigid, ...

Proposition

The category $\mathcal{M} = \mathrm{Mod}_{\mathcal{C}}(A)$ is a left $\mathcal{C}$-module category.

$$\left\{ \begin{array}{l} (M,p) \in \mathcal{M} \\ X \in \mathcal{C} \end{array} \right. \rightsquigarrow (X \cdot M, q) \in \mathcal{M} :$$

$$\begin{tikzcd}
(X \cdot M) \cdot A \arrow[rr, "a"] \arrow[dr, "q"'] & \circlearrowleft & X \cdot (M \cdot A) \arrow[dl, "\cdot p"] \\
& X \cdot M &
\end{tikzcd}$$

① $\therefore \tilde{\otimes} : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$.

② $\tilde{m}_{X,Y,M} = a_{X,Y,M}$, $\quad \tilde{\ell}_M = \ell_M$

$\varphi \in H_A(M,N) \subset H_{\mathcal{C}}(M,N)$

$$\begin{tikzcd}
M \cdot A \arrow[r, "\varphi \cdot \mathrm{id}"] \arrow[d, "p"'] & N \cdot A \arrow[d, "q"] \\
M \arrow[r, "\varphi"] & N
\end{tikzcd}$$

$$\begin{tikzcd}
1 \cdot (M \cdot A) \arrow[dr] \arrow[ddr, bend right, "\#"'] \arrow[drr, bend left] & & \\
& (1 \cdot M) \cdot A \arrow[r, "\ell_M \cdot \mathrm{id}"] \arrow[d] & M \cdot A \arrow[d] \\
& (1 \cdot M) \arrow[r, "\tilde{\ell}_M"] \arrow[ur, phantom, "\circlearrowleft"] & M
\end{tikzcd}$$

$\#$ $\quad 1$

Properties of Algebras

Proposition (adjointness of $X \mapsto X \otimes A$ and $\mathrm{Forg} : \mathrm{Mod}_{\mathcal{C}}(A) \to \mathcal{C}$)

For any $X \in \mathcal{C}$, $M \in \mathrm{Mod}_{\mathcal{C}}(A)$ there is a natural isomorphism $\mathrm{Hom}_A(X \otimes A, M) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Forg}(M))$.

$$\begin{tikzcd} H_A(X \cdot A, M) \arrow[rr, bend left, "\phi"] & & H_{\mathcal{C}}(X, M) \arrow[ll, bend left, "\psi"] \end{tikzcd}$$

$$f \in H_A(X \cdot A, M), \qquad g \in H_{\mathcal{C}}(X, M)$$

$$\phi(f) = \left(X \simeq X \cdot 1 \xrightarrow{\cdot u} X \cdot A \xrightarrow{f} M \right)$$

$$\psi(g) = \left(X \cdot A \xrightarrow{g \cdot} M \cdot A \xrightarrow{\rho} M \right)$$

Check: $\psi(g) \in H_A$, $\phi \circ \psi = \mathrm{id}$, $\psi \circ \phi = \mathrm{id}$.

Properties of Algebras

Proposition

For any $M \in \mathrm{Mod}_{\mathcal{C}}(A)$ there is a surjection $X \otimes A \to M$ for some $X \in \mathcal{C}$.

$$\begin{tikzcd} M\cdot 1 \arrow[r, "r_M"] \arrow[d, "\cdot u"'] & M \arrow[d, equal] \\ M\cdot A \arrow[r, "p"] & M \end{tikzcd}$$

$\Rightarrow$ $(id\cdot u)\circ p$ is iso

$\Rightarrow$ $M\cdot A \overset{p}{\twoheadrightarrow} M$ epi.

Proposition

If $\mathcal{C}$ has enough projectives then $\mathrm{Mod}_{\mathcal{C}}(A)$ has enough projectives.

$P(M)\cdot A$ $\in \mathrm{Mod}_{\mathcal{C}}(A)$

$$\begin{tikzcd} P(M)\cdot A \arrow[d, two heads] \arrow[dr, two heads] & \\ P(M) \arrow[r, two heads] & M \end{tikzcd}$$

$$\mathrm{Hom}_A(P(M)\cdot A, -) \cong \mathrm{Hom}_{\mathcal{C}}(P(M), -)$$

Further definitions

Definition

Two algebras A, B in $\mathcal{C}$ are *Morita equivalent* if $\mathrm{Mod}_{\mathcal{C}}(A) \cong \mathrm{Mod}_{\mathcal{C}}(B)$.

Definition

An algebra A in $\mathcal{C}$ is called *exact* if $\mathrm{Mod}_{\mathcal{C}}(A)$ is exact.

Definition

Let (M, p) and (N, q) right and left A-mods. Then $\underline{M \otimes_A N}$ is the coeq of

$$\begin{tikzcd} M \bullet A \bullet N \arrow[r, shift left, "(p \bullet id) = f"] \arrow[r, shift right, "(id \bullet q) = g"'] & M \bullet N \arrow[r, "\pi"] & M \bullet_A N \end{tikzcd}$$

$$(\pi \circ f = \pi \circ g)$$

Properties of the tensor over A

Proposition

Let (M, p) and (N, q) right and left A-mods. Then $M \otimes_A A \cong M$ and $A \otimes_A N \cong N$. Further, the functor $- \otimes_A -$ is bi-right exact.

$$\begin{tikzcd} & M \bullet 1 \arrow[r, "r_M"] \arrow[d, "\bullet u"] & M \arrow[d, equal] \\ M \bullet A \bullet A \arrow[r, shift left, "p\bullet"] \arrow[r, shift right, "\bullet m"'] & M \bullet A \arrow[r, "p"] \arrow[dr, "\pi"'] & M \arrow[d, bend left, "\psi"] \\ & & M \otimes_A A \arrow[u, "\varphi"] \end{tikzcd}$$

$\psi = \pi \circ u \circ r_M^{-1}$

$\therefore \varphi\psi = \varphi \pi u r_M^{-1}$
$= p u r_M^{-1}$
$= \mathrm{id}$

Proposition

f

If M, N right A-mods, then $\mathrm{Hom}_{\mathcal{C}}(M \otimes_A {}^*N, X) \cong \mathrm{Hom}_A(M, X \otimes N)$.

$$\begin{tikzcd} M \arrow[rrrr] \arrow[d, "\cong"'] & & & & X \bullet N \\ M \bullet 1 \arrow[r] & M \bullet ({}^*N \bullet N) \arrow[r, "\simeq"] & (M \bullet {}^*N) \bullet N \arrow[r, "\pi \bullet 1"] & (M \otimes_A {}^*N) \bullet N \arrow[ur, "f \bullet \mathrm{id}"'] & \end{tikzcd}$$

Outline

Fix $\mathcal{C}$ finite tensor cat, $\mathcal{M}$ a $\mathcal{C}$-module cat, $M_1, M_2 \in \mathcal{M}$

$$\mathcal{C} \to \mathrm{Vec}, \qquad X \mapsto \mathrm{Hom}_{\mathcal{M}}(X \otimes M_1, M_2).$$

Definition

$=: \underline{H}_{1,2}$

The object $\underline{\mathrm{Hom}}(M_1, M_2) \in \mathcal{C}$ representing the above functor is called *the internal Hom* from M_1 to M_2.

Proposition

The assignment $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{C}$, with $(M_1, M_2) \mapsto \underline{\mathrm{Hom}}(M_1, M_2)$ is (bi)-functorial. Further, $\underline{\mathrm{Hom}}$ is left exact in both variables.

Fix M_1, $f: M_2 \longrightarrow M_2'$: define $f^! = \underline{H}(M_1, -)(f)$

$$\begin{array}{l} H_{\mathcal{M}}(\underline{H}_{1,2} \cdot M_1, M_2) \simeq H_{\mathcal{C}}(\underline{H}_{12}, \underline{H}_{1,2}) \ni \mathrm{id} \\ \quad \downarrow f^{\#} \\ H_{\mathcal{M}}(\underline{H}_{1,2} \cdot M_1, M_2') \simeq H_{\mathcal{C}}(\underline{H}_{12}, \underline{H}_{1,2}') \end{array}$$

Proposition

There are natural isomorphisms

(def)
$$\mathrm{Hom}_{\mathcal{M}}(X \otimes M_1, M_2) \cong \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M_1, M_2)) \quad (1)$$

$$\mathrm{Hom}_{\mathcal{M}}(M_1, X \otimes M_2) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X \otimes \underline{\mathrm{Hom}}(M_1, M_2)) \quad (2)$$

$$\rightarrow \quad \underline{\mathrm{Hom}}(M_1, X \otimes M_2) \cong X \otimes \underline{\mathrm{Hom}}(M_1, M_2) \quad (3)$$

$$\underline{\mathrm{Hom}}(X \otimes M_1, M_2) \cong \underline{\mathrm{Hom}}(M_1, M_2) \otimes X^* \quad (4)$$

(3) $$H_{\mathcal{C}}(Y, \underline{\mathrm{Hom}}(M_1, X \cdot M_2)) \overset{(1)}{\simeq} H_{\mathcal{M}}(Y \cdot M_1, X \cdot M_2)$$
$$\simeq H_{\mathcal{M}}(X^* \cdot (Y \cdot M_1), M_2)$$
$$\overset{(1)}{\simeq} H_{\mathcal{C}}(X^* \cdot Y, \underline{H}_{1,2})$$
$$\simeq H_{\mathcal{C}}(Y, X \cdot \underline{H}_{1,2})$$

Proposition

If $M \in \mathcal{M}$ then $F_M : \mathcal{M} \to \mathcal{C}$ with $N \mapsto \underline{\mathrm{Hom}}(M, N)$ is a $\mathcal{C}$-mod functor.

$$s_{X,N} : F_M(X \cdot N) = \underline{\mathrm{Hom}}(M, X \cdot N) \simeq X \cdot F_M(N) \quad (3)$$

Corollary

If $\mathcal{M}$ is exact, then the bifunctor $\underline{\mathrm{Hom}}$ is exact.

Follow from "Sigi" and above

Proposition

(1) If $\underline{\mathrm{Hom}}$ is exact in the second variable then $\mathcal{M}$ is exact.

(2) $\mathcal{M}_1, \mathcal{M}_2$ nonzero module cats and any $F : \mathcal{M}_1 \to \mathcal{M}_2$ is exact. Then, $\mathcal{M}_1$ is exact.

X-mas

(1) P proj in $\mathcal{C}$ $\Rightarrow$ $H_{\mathcal{M}}(P \circ M, -) \simeq H_{\mathcal{C}}(P, \underline{\mathrm{Hom}}(M, -))$

$\Rightarrow$ $H_{\mathcal{M}}(P \circ M, -)$ is exact.

($\mathcal{M}$ exact iff $P \in \mathcal{C}$ proj $\Rightarrow P \otimes M \in \mathcal{M}$ proj, for all $M \in \mathcal{M}$)

(2) Claim: $\forall F \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{C}) \Rightarrow F$ is exac

Pf: $0 \neq M$ $\Rightarrow$ $F(-) \circ M$ is exact

("Sigi": $- \circ M$ exact, $X \circ M = 0 \Rightarrow X = 0$) $\Rightarrow$ F exact

$\therefore$ $\underline{\mathrm{Hom}}(N, -)$ is exact, (1) $\Rightarrow$ (2).

What does $\underline{\mathrm{Hom}}$ look like in $\mathrm{Mod}_{\mathcal{C}}(A)$?

Proposition

$A_M := \underline{\mathrm{Hom}}(M, M)$ is an algebra in $\mathcal{C}$ for all $M \in \mathcal{M}$ and $\underline{\mathrm{Hom}}(M, N)$ is a right A-module.

Recall $\phi_X : \mathrm{Hom}_{\mathcal{C}}(X, \underline{H}_{1,2}) \cong \underline{\mathrm{Hom}_{\mathcal{M}}(X \otimes M_1, M_2)}$

$X = \underline{H}_{1,2} \rightsquigarrow ev_{12} := \varphi(id) \rightsquigarrow u \quad i=2.$

$f : (\underline{H}_{2,3} \cdot \underline{H}_{1,2}) \cdot M_1 \simeq \underline{H}_{2,3} \cdot (\underline{H}_{1,2} \cdot M_1)$

$\xrightarrow[e_{12}]{\sim} \underline{H}_{2,3} \cdot M_2 \xrightarrow{e_{23}} M_3 \qquad 1=2=3 \rightsquigarrow M$

$\rightsquigarrow \phi^{-1}(f) : \underline{H}_{2,3} \cdot \underline{H}_{1,2} \longrightarrow \underline{H}_{2,3} \qquad 1=2 \;\; p:$

Outline

Assume: $\mathcal{C}$ finite (multi)tensor category, $\mathcal{M}$ a $\mathcal{C}$-module category and $M \in \mathcal{M}$ such that

(1) $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$ is right exact

(2) For any $N \in \mathcal{M}$ there is $X \in \mathcal{C}$ and a surjection $X \otimes M \to N$.

Theorem

The functor $F_M : \mathcal{M} \to \mathrm{Mod}_{\mathcal{C}}(A_M)$ given by

$$F_M(N) = \underline{\mathrm{Hom}}(M, N)$$

is an equivalence of cats.