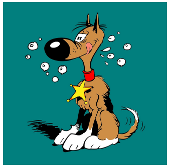
Categorification

	
set	category
morphisms	functors
equalities	isomorphic
abelian groups	abelian category
monoid	monoidal category
$x \cdot y$	$x \otimes y$
$(xy)z = x(yz)$	$a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$
$x \to 1 \cdot x$ bijection	$X \to 1 \otimes X$ equivalence
ring R	tensor category C
M: left R-module	(left) module category M
$R \times M \to M$	$C \times M \to M$
$(x, m) \mapsto xm$	$(X, M) \mapsto X \otimes M$
$(xy)m = x(ym)$	$m_{x,y,M} \quad (X \otimes Y) \otimes M \simeq X \otimes (Y \otimes M)$
$m \xrightarrow{\sim} 1 \cdot m$	$X \to 1 \otimes X$ equivalence

Module category

Let $\mathcal{C}$ be a (multi)tensor category over $\mathbf{k}$.

. $\mathcal{M}$ loc fin, abelian.

A module category $\mathcal{M}$:

- $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$: bifunctor
  - bilinear on morphisms
  - exact in first variable
  - exact in second variable → property

- natural isomorphism

$$m_{X,Y,M} : (X \otimes Y) \otimes M \simeq X \otimes (Y \otimes M) \quad \text{(module associativity constraint)}$$

such that

- $M \to 1 \otimes M$ is an auto equivalence.

- pentagon diagram:

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \otimes M \arrow[dl, "a_{X,Y,Z} \otimes id_M"'] \arrow[dr, "m_{X \otimes Y, Z, M}"] & \\
(X \otimes (Y \otimes Z)) \otimes M \arrow[d, "m_{X, Y \otimes Z, M}"] & \circlearrowleft & (X \otimes Y) \otimes (Z \otimes M) \arrow[d, "m_{X,Y,Z \otimes M}"] \\
X \otimes ((Y \otimes Z) \otimes M) \arrow[rr, "id_X \otimes m_{Y,Z,M}"'] & & X \otimes (Y \otimes (Z \otimes M))
\end{tikzcd}$$

Examples

- $\underline{R}$ is a left $\underline{R}$-module:

$\rightsquigarrow$ $\underline{\mathcal{C}}$ is a left $\underline{\mathcal{C}}$-module category:

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

$$X, Y \mapsto X \otimes Y$$

$$m_{X,Y,M} = a_{X,Y,Z}$$

- $\mathcal{C} = \mathrm{Vec}$ and $\mathcal{M} = \mathbf{Vec}$

$$X \otimes M = X \otimes^{\mathbf{Vec}} M$$

$$m_{X,Y,M} = \alpha^{\mathbf{Vec}}_{X,Y,M}$$

- Generally if $\mathcal{C} \subset \mathcal{D}$ is a (multi)tensor subcategory, then $\mathcal{D}$ is a $\mathcal{C}$-module category.

Examples

Let F : C → D be a tensor functor.
Then M = D has a structure of a module category over C with
X ⊗ Y := F (X) ⊗ Y .

$C \times D \to D$

$$m_{X,Y,M} : (X \otimes Y) \otimes M = F(X \otimes_C Y) \otimes_D M \xrightarrow{J^{-1}_{X,Y}} (F(X) \otimes_D F(Y)) \otimes M$$

$$\xrightarrow{\alpha^{\otimes}_{F(X),F(Y),M}} F(X) \otimes (F(Y) \otimes M) = X \underline{\otimes} (Y \otimes M)$$

Alternative definition

$l_M : 1 \otimes M \longrightarrow M$ unit constraint

defined by:

$$\begin{tikzcd}
1 \otimes (1 \otimes M) \arrow[rr, "L_1(l_M)"] \arrow[dr, "\alpha^{-1}_{1,1,M}"'] & & 1 \otimes M \\
& \underbrace{(1 \otimes 1)}_{c} \otimes M \arrow[ur, "\iota \otimes id_M"'] &
\end{tikzcd}$$

Prop: $L_1(l_M) = l_{1 \otimes M}$

Prop:

$$\begin{tikzcd}
(X \otimes 1) \otimes M \arrow[rr, "m_{X,1,M}"] \arrow[dr, "r_X \otimes id_M"'] & \circlearrowleft & X \otimes (1 \otimes M) \arrow[dl, "id_X \otimes l_M"] \\
& X \otimes M &
\end{tikzcd}$$

Triangle diagram.

Def $(\mathcal{M}, \otimes, m_{-,-,-}, l)$ satisfying

- pentagon diagram
- triangle diagram

is a module category.

A module is a representation

R-module $\longleftrightarrow$ R representation.

Proposition 7.3.3. There is a bijection between structures of a C-module category on M and tensor functors $F : C \to \mathrm{End}_\ell(M)$

F left exact endofunctors

Proof

Let $F : C \to \mathrm{End}_\ell(M)$

Define $X \otimes M := F(X) M$

and set

$m_{X,Y,M} : (X \otimes Y) \otimes M = F(X \otimes Y) M \xrightarrow{J_{X,Y}^{-1}} F(X)\big(F(Y) M\big) = X \otimes (Y \otimes M)$

and conversely if M is a module category over C.

then $F : C \to \mathrm{End}_\ell(M)$

$F(X) M = X \otimes M$

- Hexagon diagram $\longleftrightarrow$ pentagon diagram (since $\mathrm{End}_\ell(M)$ strict)
- $F(\mathrm{id}_C) \cong \mathrm{id}_M \longleftrightarrow 1 \otimes M \cong M$
- F exact, k-linear, F(x) is left exact $\longleftrightarrow$ $\otimes$ bilinear on morphisms exact

Example

$A = k$

$\mathcal{M} = \mathrm{Vec}$ Obviously $\mathrm{End}_{\mathcal{C}}(\mathrm{Vec}) = \mathrm{Vec}$ (Prop 2.5.4)

Thus $F : \mathcal{C} \to \mathrm{Vec}$ are in bijection with structure of module categories on Vec.

$\Rightarrow$ Fiber functors

$\Rightarrow$ theory of module categories extends the theory of fiber functors

Proposition 7.1.6. Let C be a rigid monoidal category and let M be a C-module category. There is a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{M}}(X^* \otimes M, N) \cong \mathrm{Hom}_{\mathcal{M}}(M, X \otimes N)$$

natural in X ∈ C and M, N ∈ M.

Proof

Corollary

$M \mapsto X^* \otimes M$ is left adjoint to $M \to X \otimes M$

$M \mapsto {}^*X \otimes M$ is right adjoint to $M \to X \otimes M$

$\Rightarrow X \otimes -$ is exact.

Subcategory and duals

A module subcategory N of a C-module category M is a full subcategory N ⊂ M which is closed under the action of C.

A right C-module category is the same thing as a left C^{op}-module category.

Let C be a rigid monoidal category, and let M be a right C-module category. Let $M^\vee$ be the category dual to M. Then $M^\vee$ is a left C-module category with the C-action

$$\odot : C \times M^\vee \to M^\vee$$

$$X \odot M = M \otimes {}^*X$$

and the associativity constraint given by

$$(X \otimes Y) \odot M = M \otimes {}^*(X \otimes Y) \cong M \otimes ({}^*Y \otimes {}^*X) \xrightarrow{m_{M,{}^*Y,{}^*X}} (M \otimes {}^*Y) \otimes {}^*X$$

$$= X \odot (Y \odot M)$$

Similarly, if N is a left C-module category, then $N^\vee$ is a right C-module category, with the C-action

$$N \odot X = X^* \otimes N$$

Bimodules

Let C, D be monoidal categories. A (C, D)-bimodule category is a category M that has left C-module and right D-module category structures with modules associativity constraints

$$m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M)$$

$$n_{M,W,Z} : M \otimes (W \otimes Z) \xrightarrow{\sim} (M \otimes W) \otimes Z$$

compatible by a collection

$$b_{X,M,Z} : (X \otimes M) \otimes Z \xrightarrow{\sim} X \otimes (M \otimes Z)$$

called middle associativity constraints such that the expected diagrams commute

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes M) \otimes Z \arrow[dl, "m_{X,Y,M} \otimes id_Z"'] \arrow[dr, "b_{X \otimes Y, M, Z}"] & \\
(X \otimes (Y \otimes M)) \otimes Z \arrow[d, "b_{X, Y \otimes M, Z}"] & & (X \otimes Y) \otimes (M \otimes Z) \arrow[d, "m_{X,Y,M \otimes Z}"] \\
X \otimes ((Y \otimes M) \otimes Z) \arrow[rr, "id_X \otimes b_{Y,M,Z}"'] & & X \otimes (Y \otimes (M \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \otimes (M \otimes (W \otimes Z)) \arrow[dl, "id_X \otimes n_{M,W,Z}"'] & \\
X \otimes ((M \otimes W) \otimes Z) \arrow[d, "b_{Y, M \otimes W, Z}"] & & (X \otimes M) \otimes (W \otimes Z) \arrow[ul, "b_{X,M,W \otimes Z}"'] \arrow[d, "n_{X \otimes M, W, Z}"] \\
(X \otimes (M \otimes W)) \otimes Z & & ((X \otimes M) \otimes W) \otimes Z \arrow[ll, "b_{X,M,W} \otimes id_Z"]
\end{tikzcd}$$

Module functors

morphism $\longrightarrow$ functor

$f(xm) = x f(m)$ $\qquad\qquad$ $F(X \otimes M) \simeq X \otimes F(M)$

Let M and N be two module categories over C with associativity constraints m and n, respectively.
A C-module functor from M to N consists of a functor F : M → N and a natural isomorphism

$$\eta_{X,M} : \quad F(X \otimes M) \simeq X \otimes F(M)$$

such that the following diagrams

$$\begin{tikzcd}
 & F((X \otimes Y) \otimes M) \arrow[dl, "F(m_{X,Y,M})"'] \arrow[dr, "\eta_{X \otimes Y, M}"] & \\
F(X \otimes (Y \otimes M)) \arrow[d, "\eta_{X, Y \otimes M}"'] & & (X \otimes Y) \otimes F(M) \arrow[d, "n_{X,Y,F(M)}"] \\
X \otimes F(Y \otimes M) \arrow[rr, "\mathrm{id}_X \otimes \eta_{Y,M}"'] & & X \otimes (Y \otimes F(M))
\end{tikzcd}
\qquad
\begin{tikzcd}
F(1 \otimes M) \arrow[rr, "\eta_{1,M}"] \arrow[dr, "F(l_M)"'] & & 1 \otimes F(M) \arrow[dl, "l'_{F(M)}"] \\
 & F(M) &
\end{tikzcd}$$

Exact module categories

Motivation:

$\mathcal{C} = \mathrm{Vec}$ $\quad V \otimes \mathcal{M} \cong k^n \otimes \mathcal{M} \cong \oplus (k \otimes \mathcal{M}) \cong \oplus \mathcal{M}$

There is a unique Vec module structure on $\mathcal{M}$

Understanding locally finite module category over vec.

is equivalent to understanding k-linear abelian categories

$\rightarrow$ to complicated

Let C be a multitensor category with enough projective objects. A locally finite module category M over C is called exact if for any projective object P ∈ C and any object M ∈ M the object $\boxed{P \otimes M}$ is projective in M.

Exercise 7.5.2. Let M be an arbitrary module category over C. Show that for any object X ∈ C and any projective object Q ∈ M the object X ⊗ Q is projective in M.

$\mathrm{Hom}_{\mathcal{M}}(X \otimes Q, -)$ is exact

$\cong$

$\mathrm{Hom}_{\mathcal{M}}(Q, {}^*X \otimes -)$

${}^*X \otimes$ is exact

$\mathrm{Hom}(Q, -)$ is exact

Examples

- Any semisimple module category is exact (since any object in a semisimple category is projective)

$1 \otimes M \cong M$

- Notice that in the category $C = Vec$ the object 1 is projective. Therefore for an exact module category M over C any object $M = 1 \otimes M$ is projective. Hence an abelian category M considered as a module category over C is exact if and only if it is semisimple.

- If C is semisimple (and hence 1 is projective) then any exact module category over C is semisimple.

- Any finite multitensor category C considered as a module category over itself is exact.

$\underset{\nearrow}{P} \otimes_{C} \underset{\uparrow}{X}$ is projective.

- Let C and D be finite multitensor categories and let $F : C \to D$ be a surjective tensor functor. Then the category D considered as a module category over C is exact.

$\underset{C}{X} \otimes \underset{D}{M} = F(X) \otimes_D M$

F maps proj objects to proj object

Properties of exact modules

Let M be an exact C-module category. Then the category M has enough projective objects.

Corollary: Assume that an exact module category M over C has finitely many isomorphism classes of simple objects. Then M is finite.

Proof Let P_0 is the proj cover of $\mathbb{1}$ in $\mathcal{C}$.

Then $\underbrace{P_0 \otimes X}$ $\twoheadrightarrow \mathbb{1} \otimes X \simeq X$ $\otimes$ is exact

is proj by def exact module category

Properties of exact modules

Let M be an exact module category over C. Let P ∈ C be projective and X ∈ M. Then P ⊗ X is injective.

Corollary: In an exact C-module category any projective object is injective, and vice versa.

Proof: of corollary

X is proj then $P_0 \otimes X \twoheadrightarrow 1 \otimes X \cong X$

X is a direct summand of $P_0 \otimes X$ ⇒ X is injective.

Proof $\boxed{P \otimes X}$ is injective

⟸ $Hom(-, P \otimes X)$ is exact

$\cong$

$Hom(P^* \otimes \ldots, X)$ is exact

P^* is proj. Thus $0 \to Y_1 \to Y_2 \to Y_3 \to 0$ exact

$0 \to P^* \otimes Y_1 \to P^* \otimes Y_2 \to P^* \otimes Y_3 \to 0$ splits

Complete reducibility of exact module categories

Let $\mathcal{O}(\mathcal{M})$ denote the set of (isomorphism classes of) simple objects in M
Let us introduce the following relation on $\mathcal{O}(\mathcal{M})$:
two objects X, Y ∈ $\mathcal{O}(\mathcal{M})$ are related if Y appears as a subquotient of
L ⊗ X for some L ∈ C.

Lemma: The relation above is reflexive, symmetric and transitive.

$\to$ equivalence classes $\mathcal{O}(\mathcal{M}) = \bigsqcup_{i \in I} \mathcal{O}(\mathcal{M}_i)$

Let M_i denote the full subcategory of M consisting of objects whose simple subquotients lie in the .
$\mathcal{O}(\mathcal{M})_i$

Prop
The module categories M_i are exact. The category M is
the direct sum of its module subcategories M_i i.e.,

$$\mathcal{M} = \bigoplus_{i \in I} \mathcal{M}_i$$

$P_0 \twoheadrightarrow 1$

If $X \in \mathcal{O}(\mathcal{M}_i)$ then $P_0 \otimes X \in \mathcal{M}_i$

$P_0 \otimes X \twoheadrightarrow 1 \otimes X$

proj cover of X is contained in $\mathcal{M}_i$

$$\Rightarrow \mathcal{M} = \bigoplus_{i \in I} \mathcal{M}_i$$

characterizing property of exact module categories

Let M_1 and M_2 be two module categories over C. Assume that M_1 is exact.
Then any additive module functor F : M_1 → M_2 is exact.

Proof

$0 \to X \to Y \to Z \to 0$ exact in M_1

Assume

$0 \to F(X) \to F(Y) \to F(Z) \to 0$ is not exact

Then

$\oplus$ is exact

$0 \to P \otimes F(X) \to P \otimes F(Y) \to P \otimes F(Z) \to 0$ not exact $\quad$ [$P \otimes X = 0$ then $X = 0$]

But $\quad 0 \to P \otimes X \to P \otimes Y \to P \otimes Z \to 0$ $\quad$ P is proj
exact and splits

$0 \to F(P \otimes X) \to F(P \otimes Y) \to F(P \otimes Z) \to 0$ $\quad$ split exact $\lightning$.