Chapter 5 part I

Kleine seminar

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Four subsections

n

1. (Quasi-)Fiber fuctor

2. Bialgebras

3. Hopf algebras

4. Reconstruction theory in the infinite setting

(Quasi-)Fiber fuctor

What is a fiber functor

Let C be a ring category over $\mathbb{k}$.

locally fin $\mathbb{k}$-linear abelian monoidal category

with $\otimes$ bilinear biexact

multiring category

$+ \quad \mathrm{End}\, 1 \simeq \mathbb{k}$ is a ring category

What is a fiber functor

Let $\mathcal{C}$ be a ring category over $\mathbb{k}$.

Definition

A quasi-fiber functor on $\mathcal{C}$ is an exact faithful functor

$$F : \mathcal{C} \to \mathrm{Vec}, \qquad F(1) = \mathbb{k},$$

with a natural isomorphism $J : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y), \quad X, Y \in \mathcal{C}$. It is called a fiber functor if J is a tensor structure.

$$\begin{tikzcd}
(F X \otimes F Y) \otimes F(Z) \arrow[rr] \arrow[d, "J"] & & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "J"] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "J"] & & F(X) \otimes F(Y \otimes Z) \arrow[d] \\
F((X \otimes Y) \otimes Z) \arrow[rr] & & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

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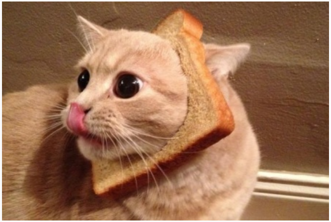

$$\begin{tikzcd}
(FX\,FY)\,FZ \arrow[rr] \arrow[d] & & FX\,(FY\,FZ) \arrow[d] \\
FXY\,FZ \arrow[d] & \circlearrowright & FX\,FYZ \\
F(XY)Z \arrow[rr] & & FX(YZ)
\end{tikzcd}$$

Examples and Nonxamples

An example to remember

The forgetful functors $\mathrm{Vec}_G \to \mathrm{Vec}$ and $\mathrm{Rep}(G) \to \mathrm{Vec}$ are fiber functor.

A non-example

If $\omega \in Z^3(G, \Bbbk^\times)$ is a cohomologically non-trivial 3-cocycle, then the forgetful functor $\mathrm{Vec}_G^\omega \to \mathrm{Vec}$ is a quasi-fiber functor but Vec_G^ω does not admit a fiber functor.

2.6

if J was a Tensor structure then ω on Vec_G^ω was equivalent cohomologically to id

(2.31)

Bialgebras

Deligne's tensor product

- $\boxtimes$ is right exact bifunctor in both variables, universal

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{D} \arrow[r, "\boxtimes"] \arrow[d, "F"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C} \boxtimes \mathcal{D} \arrow[dl, dashed, "\bar{F}"] \\
\mathcal{A} &
\end{tikzcd}$$

F bifunctor right exact in both var.

Prop 1.11.2

iii) C, D coalgebras C-comod $\boxtimes$ D-comod $=$ $(C \otimes D)$-comod

iv) $\mathrm{Hom}_{\mathcal{C}}(X_1, Y_1) \otimes \mathrm{Hom}_{\mathcal{D}}(X_2, Y_2) \simeq \mathrm{Hom}_{\mathcal{C} \boxtimes \mathcal{D}}(X_1 \boxtimes X_2, Y_1 \boxtimes Y_2)$

Structures on endomorphism

category → Fiber functor

Let $H = \mathrm{End}(F)$. Recall $\alpha : \mathrm{End}(F) \otimes \mathrm{End}(F) \xrightarrow{\sim} \mathrm{End}(F \boxtimes F)$. ← IV)

Coproduct and counit

Let $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \Bbbk$ defined by

$$\Delta(a) = \alpha^{-1}(\tilde{\Delta}(a)) = \alpha^{-1}(J^{-1}aJ)$$

$$\varepsilon(a) = a_1 \in \Bbbk$$

Theorem 5.2.1

i) The algebra H is a coalgebra with coproduct Δ and counit ε.

ii) The maps Δ and ε are unital algebra homomorphisms.

i) J is a tensor structure → coassociativity

ii) $\Delta(a)\Delta(b) \rightsquigarrow \alpha^{-1}(J^{-1}aJ)\,\alpha^{-1}(J^{-1}bJ) \rightsquigarrow \alpha^{-1}(J^{-1}aJJ^{-1}bJ)$

$\Delta(ab) \rightsquigarrow \alpha^{-1}(J^{-1}abJ) \longleftarrow =$

What is a bialgebra

Bialgebra

An algebra H is a bialgebra if it has a coproduct Δ and counit ε respecting

1. H is a coalgebra with coproduct Δ and counit ε.
2. The maps Δ and ε are unital algebra homomorphisms.

$\mathcal{H} = \mathrm{End}_{\mathcal{C}}(F)$ for F fiber functor,

$\mathcal{H}$ is a bialgebra

Monoidal categories from bialgebras

Module categories

Let H be a bialgebra. Then its left modules $\mathbf{Rep}(H)$ and its finite-dimensional left-modules $\mathrm{Rep}(H)$ are monoidal categories. The action of H is

$$\rho_{X\otimes Y}(a) = \rho_X \otimes \rho_Y(\Delta(a)), \quad \rho_X : H \longrightarrow \mathrm{End}(X),\ \rho_Y : H \longrightarrow \mathrm{End}(Y)$$

and the unit object is $\Bbbk$ upon which H acts by $a \longrightarrow \varepsilon(a)$.

Forget it not

The forgetful functor $\mathrm{Forget} : \mathrm{Rep}(H) \longrightarrow \mathrm{Vec}$ is a fiber functor.

Monoidal categories from bialgebras

Right module categories

Let H be a bialgebra. Then its right modules $H - \mathbf{comod}$ and its finite-dimensional right-modules $H - \mathrm{comod}$ are monoidal categories. The coaction of H is

$$\pi_{X\otimes Y}(x\otimes y) = \sum x_i \otimes y_j \otimes a_j b_j, \qquad \pi_X(x) = \sum x_i \otimes a_i,\ \pi_Y(y) y_i \otimes b_j.$$

Theorems

Theorem 5.2.3 ***The reconstruction theorem for fin-dim biaglebras***

The assignments

$$(\mathcal{C}, F) \mapsto H = \mathrm{End}_{\mathcal{C}}(F) \quad H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$$

are mutually inverse bijections between

1. finite ring categories $\mathcal{C}$ with fiber functor F (up to tensor equivalence and iso of tensor functors)
2. isomorphism classes of finite dimensional $\mathbb{k}$-bialgebras H

Stranger:

this an equivalence of cat

FBialg ——

→ FRCF

$H \longmapsto \mathrm{Rep}H$

5.2.5

Show that the axioms of a bialgebra are self-dual in the following sense: if H is a finite dimensional bialgebra with multiplication $\mu : H \otimes H \longrightarrow H$, unit $i : \mathbb{k} \longrightarrow H$, comultiplication $\Delta : H \longrightarrow H \otimes H$ and counit $\varepsilon : H \longrightarrow \mathbb{k}$, then H^* is also a bialgebra, with the multiplication Δ^*, unit ε^*, comultiplication μ^*, and counit i^*.

$$\begin{tikzcd} & H \otimes H \arrow[dr, "\mu"] & \\ H \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] & K \arrow[r, "i"] & H \end{tikzcd}$$

Dual $\rightsquigarrow$

$$\begin{tikzcd} & H^* \otimes H^* \arrow[dl, "\Delta^*"'] & \\ H^* & K^* \arrow[l, "\varepsilon^*"'] & H^* \arrow[l, "i^*"'] \arrow[ul, "\mu^*"'] \end{tikzcd}$$

$\oplus$ Proper rel

Hopf algebras

Goal of the book

Prove that the bialgebra $H = \mathrm{End}(F)$ has a Hopf algebra structure before defining what a Hopf algebra is.

What is a Hopf algebra

Antipode

Let H be a bialgebra. An antipode $S : H \to H$ is a linear map respecting

antipod. rel. $$\mu \circ (\mathrm{id} \otimes S) \circ \Delta = i \circ \varepsilon = \mu \circ (S \otimes \mathrm{id}) \circ \Delta.$$

A bialgebra with invertible antipode S is a Hopf algebra.

+ S invertible

$$\begin{tikzcd}
 & H \otimes H \arrow[r, "\mathrm{id} \otimes S"] & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & \mathbb{K} \arrow[rr, "i"] & & H \\
 & H \otimes H \arrow[r, "S \otimes \mathrm{id}"'] & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Antipode

Let H be a bialgebra. An antipode $S : H \to H$ is a linear map respecting

$$\mu \circ (\mathrm{id} \otimes S) \circ \Delta = i \circ \varepsilon = \mu \circ (S \otimes \mathrm{id}) \circ \Delta.$$

A bialgebra with invertible antipode S is a Hopf algebra.

For $H = \mathrm{End}(F)$

If C has left duals, then $S : H \to H$ is defined by $S(a)_X = a^*_{X^*}$.

2.10 $ev_X : X^* \otimes X \longrightarrow 1$; $coev_X : 1 \longrightarrow X \otimes X^*$

$X \xrightarrow{coev} (X \otimes X^*) \otimes X \xrightarrow{\text{associator}} X \otimes (X^* \otimes X) \xrightarrow{ev} X$ $\quad id_X$

$X^* \xrightarrow{coev} X^* \otimes (X \otimes X^*) \xrightarrow{\text{aso}} (X^* \otimes X) \otimes X^* \xrightarrow{ev} X^*$ $\quad id_{X^*}$

F fiber functor, Vec, k

2.10.6 $ev_{F(X)}$ $\quad F(X^*) \otimes F(X) \xrightarrow{J} F(X^* \otimes X) \xrightarrow{F(ev)} F(1) \xrightarrow{=} k$

$coev_{F(X)}$ $\quad 1 \longrightarrow F(1) \xrightarrow{F(coev)} F(X \otimes X^*) \xrightarrow{J^{-1}} F(X) \otimes F(X^*)$

"Fun" with diagrams

$b \in \mathrm{End}\, F \otimes F.$ $\quad b = \eta \otimes \nu$

$\boxed{\mu \circ (\mathrm{id} \otimes S) \circ \Delta(b)} = 1 \circ \varepsilon(b)$

$$F(x) \xrightarrow{\mathrm{coev}} (F(x) \otimes F(x)^*) \otimes F(x) \xrightarrow{b_{\mathrm{lmo}}} F(x) \otimes (F(x)^* \otimes F(y)) \xrightarrow{\mathrm{ev}} F(y)$$

$$\begin{tikzcd}
F(X) \arrow[r, "\mathrm{coev}"] \arrow[d, "\mathrm{id}"'] \arrow[ddd, bend right, "\mathrm{id}\,\varepsilon"'] & F(x) \otimes F(X)^* \otimes F(X) \arrow[d, "J"] \arrow[ddd, bend left] \\
F(X) \arrow[r, "F\mathrm{coev}_x"] \arrow[d, "\varepsilon(h)\,\mathrm{id}"'] & F(X \otimes X^*) \otimes F(X) \arrow[d, "b"] \\
F(X) \arrow[r, "F(\mathrm{coev}_x)"] \arrow[d, "\mathrm{id}"'] & F(X \otimes X^*) \otimes F(X) \arrow[d, "J^{-1}"] \\
F(x) & F(x) \otimes F(x)^* \otimes F(x) \arrow[l, "\mathrm{ev}"']
\end{tikzcd}$$

Antipode are anti-homomorphism

Proposition 5.3.6

If S is an antipode on a bialgebra H, then S is an anti-homomorphism of (co)algebras with (co)unit. $S(ab) = S(b)\,S(a)$

Coro. 5.3.7

1. If H is a bialgebra with antipode S, then $\mathcal{C} = \mathrm{Rep}(H)$ has left duals. For X^* the usual dual, the H-action

$$\rho_{X^*}(a) = \rho_X(S(a))^*.$$

2. If S is invertible, then $\mathcal{C}$ also admits right duals, so it is a tensor category. The right dual ${}^*X$ with H-action

$$\rho_{{}^*X}(a) = \rho_X(S^{-1}(a))^*.$$

Tensor cat: local fin k-lin. abelian rigid cat with $\otimes$ bilin End $1 \cong \mathbb{k}$

Cofun

ρ is an action

S is an anti-homo

$S(ab) = S(b)S(a)$

i) $\rho_{X^*}(ab) = \rho_X(S(ab))^*$

$= \rho_X(S(b)\,S(a))^*$

$= (\rho_X(Sb)\,\rho_X(Sa))^* = \rho_X{S(a)}^* \rho_X S(b)^*$

$= \rho_{X^*} a \; \rho_{X^*} b$

ii) is similar

Hence if H is a Hopf alg. the Rep H is a Tensor category

5.3.5 Antipodes are unique

If an antipode exists on a bialgebra, it is unique.

$S' : H \to H$ another antipode

$$
\begin{tikzcd}
 & H\otimes H \arrow[r, "\mathrm{id}\, S"] \arrow[r, bend left, "\mathrm{id}\, S'"] & H\otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & \mathbb{K} \arrow[rr, "i"] & & H \\
 & H\otimes H \arrow[r, "S\, \mathrm{id}"] \arrow[r, bend right, "S'\, \mathrm{id}"'] & H\otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

Assignments

Theorem 5.3.12

$$(\mathcal{C}, F) \mapsto H = \mathrm{End}_{\mathcal{C}}(F), \quad H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$$

are mutually inverse bijections between

1. isomorphism classes of finite tensor categories $\mathcal{C}$ with fiber functor F (up to tensor equivalence and iso of tensor functors)
2. isomorphism classes of finite dimensional Hopf $\Bbbk$-algebras.

Tensor Cat

Hopf alg

Proposition 5.3.15 Finite bialgebra with antipode are Hopf

H finite-dimensional bialgebra with $S: H \to H$, an antipode

$H_n = \operatorname{Im} S^n$ then since it is fin-dim $\exists m$
such that $H_{m+1} = H_m$

Prove that $m = 0$, $H_1 = H$,

On $H_1 = \operatorname{Im} S$ $\exists S'$ such that $S'S|_{H_1} = \mathrm{id}_{H_1}$

$a \in H$

$$H \xrightarrow{(\Delta \otimes \mathrm{id})\Delta(a)} H \otimes H \otimes H$$

if we use antipode on many way, we find
$b \in H_1$ such that $b = a$

$\mu^{op}(x \otimes y) = \mu(y \otimes x)$

5.3.17

Let $(H, \mu, i, \Delta, \varepsilon, S)$ be a Hopf algebra. Let μ^{op} and Δ^{op} be obtained by permutation of components. The following are Hopf algebra

$$H_{op} := (H, \mu^{op}, i, \Delta, \varepsilon, S^{-1}) \tag{1}$$

$$H^{cop} := (H, \mu, i, \Delta^{op}, \varepsilon, S^{-1}) \tag{2}$$

$$H^{cop}_{op} := (H, \mu^{op}, i, \Delta^{op}, \varepsilon, S) \tag{3}$$

Furthermore, $H \simeq H^{cop}_{op}$ and $H_{op} \simeq H^{cop}$, so $\mathrm{Rep}(H)^{op} \simeq \mathrm{Rep}(H_{op})$. And if H is (co)commutative, $\mu = \mu^{op}$ ($\delta = \delta^{op}$), then $S = S^{-1}$.

Results

Reconstruction theory in the infinite setting

What happens in the infinite-dimensional setting

- $\mathcal{C}$ ring category over $\Bbbk$ not finite
- Then $\mathrm{End}(F \otimes F) \simeq \mathrm{End}(F) \hat{\otimes} \mathrm{End}(F)$ (completion for inverse limit topology)
- coalgebra $\mathrm{Coend}(F)$ (Sect. 1.10)

$$\mathrm{Coend}(F) = \Big(\bigoplus_{X \in \mathcal{C}} F(X)^* \otimes F(X)\Big)/E \qquad E = \langle y_* \otimes F(f)x - F(f)^* y_* \otimes x \cancel{\langle} \rangle$$

- The same antipode works

$$S(a)_X = a^*_{X^\vee}$$

Big theorem

Theorem 5.4.1 Assignments

The assignments

$$(C, F) \mapsto H = \mathrm{Coend}(F), \qquad H \mapsto (H - \mathrm{comod}, \mathrm{Forget})$$

are mutually inverse bijections for pairs of:

- ring cat with fiber functor and bialgebras
- ring cat with left duals with fiber functor and bialgebra with antipode
- tensor cat with fiber functor and Hopf algebras

What is not working?

Non-coro

Bialgebra with antipode are not necessarly Hopf algebras (thus S is not necessarily invertible).

Examples (5.4.4)

1. $\mathcal{C}$ category of algebraic representation of an affine algebraic group over $\Bbbk$ with forgetful functor. Then $\mathrm{Coend}(F) = \mathcal{O}(G)$ is the Hopf algebra of regular functions on G
2. $\mathcal{C}$ category of finite-dimensional G-rep over $\Bbbk$ with forgetful functor $\mathrm{Coend}(F)$ is a commutative Hopf algebra.