Tensor categories

Kleine Seminar

Ghent University

4. Tensor categories
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4.7 Quantum traces, pivotal and spherical categories
4.8 Semisimple multitensor categories
4.9 Grothendieck rings of semisimple tensor categories

Deligne's tensor product of tensor categories

What is a Deligne's tensor product?

$\mathcal{C}, \mathcal{D}$, two locally finite abelian categories over a field k.

Definition

Deligne's tensor product $(\mathcal{C} \boxtimes \mathcal{D}, \boxtimes)$

right exact, bilinear

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{D} \arrow[rr, "\boxtimes"] \arrow[dr, "F"'] & & \mathcal{C} \boxtimes \mathcal{D} \arrow[dl, dotted, "\bar{F}"] \\
& \mathcal{A} &
\end{tikzcd}$$

Deligne's tensor product of tensor categories

Some properties

Proposition 1.11.2. (i) *A Deligne's tensor product $\mathcal{C} \boxtimes \mathcal{D}$ exists and is a locally finite abelian category.* ess sm

(ii) *It is unique up to a unique equivalence.*

(iii) *Let C, D be coalgebras and let $\mathcal{C} = C-\mathsf{comod}$ and $\mathcal{D} = D-\mathsf{comod}$. Then $\mathcal{C} \boxtimes \mathcal{D} = (C \otimes D)-\mathsf{comod}$.*

(iv) *The bifunctor $\boxtimes$ is exact in both variables and satisfies*

$$\mathsf{Hom}_{\mathcal{C}}(X_1, Y_1) \otimes \mathsf{Hom}_{\mathcal{D}}(X_2, Y_2) \cong \mathsf{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(X_1 \boxtimes X_2, Y_1 \boxtimes Y_2).$$

(v) *Any bilinear bifunctor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ exact in each variable defines an exact functor $\bar{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$.*

Deligne's tensor product of tensor categories

The construction of Deligne's tensor product is given by

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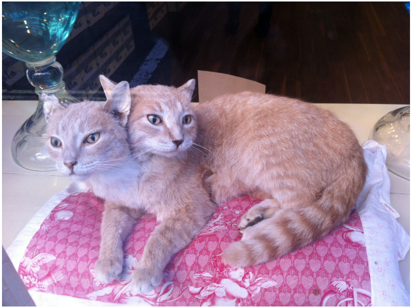
THEOREM 1.9.15 (Takeuchi, [**Tak2**]). *Any essentially small locally finite abelian category $\mathcal{C}$ over a field $\Bbbk$ is equivalent to the category $C-\mathrm{comod}$ for a unique pointed coalgebra C. In particular, if $\mathcal{C}$ is finite, it is equivalent to the category $A-\mathrm{mod}$ for a unique basic algebra A (namely, $A = C^*$).*

Proposition

Deligne's tensor product of (multi)ring categories (resp. (multi)tensor categories, resp. (multi)fusion categories) is again a (multi)ring category (resp. (multi)tensor category, resp. (multi)fusion categories).



Proof idea.

Need to define "tensor product" on $\mathcal{C} \boxtimes \mathcal{D}$

$$\otimes \quad : \quad \mathcal{C} \times \mathcal{C} \xrightarrow{\quad \boxtimes \quad} \mathcal{C} \boxtimes \mathcal{C} \xrightarrow{\quad T_{\mathcal{C}} \quad} \mathcal{C}$$

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Define for $X, Y \in \mathcal{C} \boxtimes \mathcal{D}$

$$X \otimes Y := ((T_{\mathcal{C}} \boxtimes T_{\mathcal{D}}) \circ (23))(X \boxtimes Y)).$$

Then all relations for $\otimes$ in $\mathcal{C}, \mathcal{D}$ can be transferred to this new tensor product by the universal property of $\boxtimes$.

$23 : \mathcal{C} \boxtimes \mathcal{D} \boxtimes \mathcal{C} \boxtimes \mathcal{D}$

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- multi vs single? Proposition 1.11.2(iv)
- tensor category? Duality functor $*_{\mathcal{C}} \boxtimes *_{\mathcal{D}}$
- Finiteness & semisimpleness (i.e. fusion)? Theorem 1.9.15

□

Quantum traces, pivotal and spherical categories

Definition

Let $\mathcal{C}$ be a rigid monoidal category, V be an object in $\mathcal{C}$, and $a \in \mathrm{Hom}_{\mathcal{C}}(V, V^{**})$. Define its left categorical (or quantum) trace by

$$\mathrm{Tr}^L(a) : \mathbb{1} \longrightarrow \underset{v^* \, v}{V^* \otimes V} \xrightarrow{\mathrm{id} \otimes a} V^* \otimes V^{**} \xrightarrow{ev_V} \mathbb{1}$$

and for $a \in \mathrm{Hom}_{\mathcal{C}}(V^{**}, V)$ its right quantum trace by

$$\mathrm{Tr}^R(a) :$$

Quantum traces, pivotal and spherical categories

Proposition

If $a \in \mathrm{Hom}_{\mathcal{C}}(V, V^{**}), b \in \mathrm{Hom}_{\mathcal{C}}(W, W^{**})$ *then*

1. $\mathrm{Tr}^L(a) = Tr^R(a^*)$;
2. $\mathrm{Tr}^L(a \oplus b) = Tr^L(a) + \mathrm{Tr}^R(b)$ *(in additive categories);*
3. $\mathrm{Tr}^L(a \otimes b) = Tr^L(a) Tr^L(b)$;
4. *If* $c \in Hom(V, V)$ *then*
   $\mathrm{Tr}^L(ac) = \mathrm{Tr}^L(c^{**}a), \mathrm{Tr}^R(ac) = \mathrm{Tr}^R({}^{**}ca)$.

Proposition

For $\mathcal{C}$ a multitensor category, if $a \in \mathrm{Hom}_{\mathcal{C}}(V, V^{**})$ and $W \subset V$ such that $a(W) \subset W^{**}$ then $\mathrm{Tr}^L(a) = \mathrm{Tr}^L(a|_W) + \mathrm{Tr}^L(a|_{V/W)}$.

Definition

Let $\mathcal{C}$ be a rigid monoidal category. A *pivotal structure* on $\mathcal{C}$ is an isomorphism of monoidal functors $a_X : X \to X^{**}$. We call the category $\mathcal{C}$ pivotal.

Quantum traces, pivotal and spherical categories

Definition

Let $\mathcal{C}$ be a pivotal category with pivotal structure a. The dimension of an object X is

$$\underline{\dim_a(X) = \mathrm{Tr}^L(a_X)} \in \mathrm{End}_{\mathcal{C}}(\mathbf{1}).$$

Proposition

If $\mathcal{C}$ is a tensor category, then $([X] \mapsto \dim_a(X)) \in \underline{\mathrm{Hom}_{Rings}(\mathrm{Gr}(\mathcal{C}), k)}$, i.e. it is a character of the Grothendieck ring.

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$\overline{Z[\bar{\alpha}]}$

Corollary

Dimension of objects in a pivotal finite tensor category are algebraic integers in k.

Definition

A pivotal structure a on a tensor category $\mathcal{C}$ is *spherical* if $\dim_a(V) = \dim_a(V^*)$ for any object V in $\mathcal{C}$. A spherical category is a tensor category equipped with a spherical structure.

Quantum traces, pivotal and spherical categories

Theorem

Let $\mathcal{C}$ be a spherical category and V be an object of $\mathcal{C}$. Then for any $x \in \mathrm{Hom}_{\mathcal{C}}(V, V)$ one has

$$\mathrm{Tr}^L(a_V x) = \mathrm{Tr}^R(x a_V^{-1}).$$

Proof.

First assume V semisimple. $V = \bigoplus_i Y_i \otimes V_i$, where V_i are simple objects and Y_i are vector spaces. Then $x = \oplus_i x_i \otimes \mathrm{id}_{V_i}$ and $a_V = \oplus \mathrm{id}_{Y_i} \otimes a_{V_i}$. Then

$$\mathrm{Tr}^L(a_V x) = \sum_i \mathrm{Tr}(x_i) \dim(V_i)$$

$$\mathrm{Tr}^R(x a_V^{-1}) = \sum_i \mathrm{Tr}(x_i) \dim(V_i^*)$$

For the general case, use the *socle filtration* (section 1.10)

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V,$$

a filtration where V_{i+1}/V_i is semisimple. Then

$$\mathrm{Tr}^L(a_V x) = \sum_i \mathrm{Tr}^L(a_V x|_{V_{i+1}/V_i}) = \sum_i \mathrm{Tr}^R(x a_V^{-1}|_{V_{i+1}/V_i}) = \mathrm{Tr}^R(x a_V^{-1})$$

Let k be an algebraically closed field.

Proposition

Let $\mathcal{C}$ be a semisimple multitensor category over k and let V be an object in $\mathcal{C}$. Then ${}^*V \cong V^*$. Hence, $V \cong V^{**}$.

Proof.

May assume V simple.

$\mathcal{C}$ is semisimple

$\implies \dim(\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V \otimes X)) = \dim(\mathrm{Hom}_{\mathcal{C}}(V \otimes X, \mathbf{1}))$ for any X.

But

$$\dim(\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V \otimes X) \neq 0 \iff X \cong V^*$$
$$\dim(\mathrm{Hom}_{\mathcal{C}}(V \otimes X, \mathbf{1}) \neq 0 \iff X \cong {}^*V.$$

□

Proposition

$\mathcal{C}$ be a semisimple tensor category over k, let V be a simple object in $\mathcal{C}$ and $a\colon V \xrightarrow{\sim} V^{**}$ be an isomorphism. Then $\mathrm{Tr}^R(a) \neq 0 \neq \mathrm{Tr}^L(a)$.

Proof.

the trace of a is

$$\mathbf{1} \to V \otimes V^* \to \mathbf{1},$$

with both morphisms non-zero. If the composition is zero:

$$\underbrace{V \otimes V^* \to 1}_{\neq 0}$$

1

$\dim(V, V) > 1$

□

Grothendieck rings of semisimple tensor categories

Proposition

If $\mathcal{C}$ is a semisimple multitensor category then $\mathrm{Gr}(\mathcal{C})$ is a based ring. If $\mathcal{C}$ is a semisimple tensor category then $\mathrm{Gr}(\mathcal{C})$ is a unital bades ring. If $\mathcal{C}$ is a (multi)fusion category then $\mathrm{Gr}(\mathcal{C}$ is a (multi)fusion ring.

Proof.

- $\mathbf{Z}_+$-basis? Isomorphism classes of simple objects.
- Set I_0? Simple subobjects of $\mathbf{1}$.
- Duality map becomes (anti)-involution.

$\to \tau(b_i b_j) = \{ {1 \atop 0}$

$\dim \mathrm{Hom}(V \otimes W, \mathbb{1})$

- If category is finite, then in particular $\mathrm{Gr}(\mathcal{C})$ of finite rank.

$\square$

Example (Counterexample)

Category of $\mathcal{S}_3$-reps over an algebraically closed field of characteristic 2.

$$[V \otimes V^* : \mathbf{1}] > 1$$

irrep dim 2

Example

Category of finite dimensional reps of $\mathfrak{sl}_2(\mathbb{C})$. Simple objects $V_m, m \in \mathbb{N}$ and $V_0 = \mathbf{1}$. Product on $\mathrm{Gr}(\mathcal{C})$ determined by the *Clebsch-Gordan rule*

$$V_i \otimes V_j = \bigoplus_{l=0}^{\min(i,j)} V_{i+j-2l}.$$

This is a unital based ring.

Grothendieck rings of semisimple tensor categories

$\mathcal{C}$ a semisimple multitensor categori withs imple objects $\{X_i\}_{i \in I}$. Let I_0 be the subset of I such that $\mathbf{1} = \oplus_{i \in I_0} X_i$. Let $H_{ij}^l := \mathrm{Hom}(X_l, X_i \otimes X_j)$. These determine the multiplication on $\mathrm{Gr}(\mathcal{C})$.

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Subject to restrictions!

- associativity constraint reduces to linear isomorphisms

$$\Phi^{i_4}_{i_1 i_2 i_3} : \bigoplus_j H^j_{i_1 i_2} \otimes H^{i_4}_{j i_3} \cong \bigoplus_l H^{i_4}_{i_1 l} \otimes H^l_{i_2 i_3}$$

  The matrix blocks $\left(\Phi^{i_4}_{i_1 i_2 i_3}\right)_{jl} : H^j_{i_1 i_2} \otimes H^{i_4}_{j i_3} \to H^{i_4}_{i_1 l} \otimes H^l_{i_2 i_3}$ are called 6*j-symbols*.
- Pentagon identity also turns into relation on $6j$-symbols.

In the case of finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$: H_{ij}^{l} are 0- or 1-dimensional
After choosing basis vectors: $6j$-symbols are numbers. *Racah coefficients* or *classical* $6j$-*symbols*.