# Tensor categories

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# 4.1. Tensor and multitensor categories

Let $\mathcal{C}$ be a locally finite $\Bbbk$-linear abelian rigid monoidal category.

$\exists X^*, {}^*X$

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$\mathrm{Hom}(X,Y)$ fin. dim.

$X$ finite length

$\exists \, \mathrm{Ker}(\phi), \ \mathrm{coker}(\phi)$

additive, $\mathrm{Hom}(X,Y)$ has VS structure over $\Bbbk$

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- $\mathcal{C}$ *is a* **fusion category** *if* $\mathcal{C}$ *is a multifusion category with* $\mathbf{End}_{\mathcal{C}}(\mathbf{1}) \cong \Bbbk$, *i.e,*
  *if* $\mathcal{C}$ *is a finite semisimple tensor category.*

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- If $\mathrm{char}(\Bbbk) = 0$ or $\mathrm{char}(\Bbbk)$ is coprime to $|G|$, then $\mathbf{Rep}(G)$ is a fusion category.
- $\mathbf{Rep}(\mathfrak{g})$ = category of finite dimensional representations of a Lie algebra $\mathfrak{g}$ is a tensor category.

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Objects of $\mathbf{Mat}_n(\mathbf{Vec})$ are $n \times n$ matrices of vector spaces $(V_{ij})$ and the tensor product is matrix multiplication.

$$(V \otimes W)_{il} = \bigoplus_{j=1}^{n} V_{ij} \otimes W_{jl}$$

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### Proposition

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## Proof.

Suppose $V$ is an object in $\mathcal{C}$.

- $\mathcal{C}$ rigid $\implies$ $\exists\ V^*, {}^*V$
- $\mathcal{C}$ monoidal $\overset{(2.10.8)}{\implies}$ $(V \otimes -), (- \otimes V)$ have left/right adjoint functors
- $\mathcal{C}$ abelian $\overset{(1.6.4)}{\implies}$ $(V \otimes -), (- \otimes V)$

□

# Remark

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The simple objects are $\mathbb{C}_+ = \mathbf{1}$ and $\mathbb{C}_-$, with respective bimodule structures given by

$$(a, b) \cdot z = azb \quad \text{and} \quad (a, b) \cdot z = az\bar{b}$$

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$\otimes$ is $\mathbb{R}$-bilinear, but not $\mathbb{C}$-bilinear on morphisms.

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Compared to tensor categories, we require biexactness of $\otimes$ instead the existance of duals.

# Examples of multiring and ring categories

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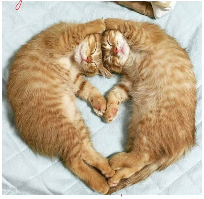
- Every (multi)tensor category is a (multi)ring category.

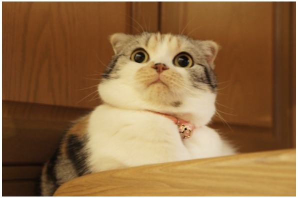
fusion cat



finite semisimple tensor cat

$\subset$

tensor cat



rigid ring cat

$\subset$

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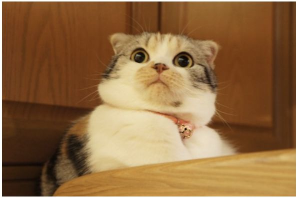
 $\subset$  $\subset$ 


- $\mathbf{Vec}_G$ = category of finite dimensional $\mathbb{k}$-vector spaces graded by a monoid $G$ is a ring category, with tensor product

$$(V \otimes W)_g = \bigoplus_{x,y \in G : xy = g} V_x \otimes W_y.$$

# Tensor functors

Let $\mathcal{C}$, $\mathcal{D}$ be multiring categories over $\Bbbk$, and let $F : \mathcal{C} \to \mathcal{D}$ be an exact and faithful $\Bbbk$-linear functor.

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## Definition

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$$J : F(-) \otimes F(-) \to F(- \otimes -).$$

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$$J : F(-) \otimes F(-) \to F(- \otimes -).$$

- $F$ *is a* **tensor functor** *if it is a monoidal quasi-tensor functor* $(F, J)$.

# Proposition

For any pair of morphisms $f_1$, $f_2$ in a multiring category $\mathcal{C}$ we have

$$\mathrm{Im}(f_1 \otimes f_2) = \mathrm{Im}(f_1) \otimes \mathrm{Im}(f_2).$$

# Proposition

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## Proof.

$0 \to X \to Y \to Z \to 0$ exact

$(T\otimes -)$

$$\Longrightarrow \quad 0 \to T \otimes X \to T \otimes Y \to T \otimes Z \to 0 \quad \text{exact}$$

$\mathrm{Hom}(-,1)$

$$\Longrightarrow \quad 0 \to \mathrm{Hom}_{\mathcal{C}}(T \otimes Z, \mathbf{1}) \to \mathrm{Hom}_{\mathcal{C}}(T \otimes Y, \mathbf{1}) \to \mathrm{Hom}_{\mathcal{C}}(T \otimes X, \mathbf{1}) \quad \text{exact}$$

$(2.10.8)$

$$\Longrightarrow \quad 0 \to \mathrm{Hom}_{\mathcal{C}}(T, Z^*) \to \mathrm{Hom}_{\mathcal{C}}(T, Y^*) \to \mathrm{Hom}_{\mathcal{C}}(T, X^*)$$

$$\Longrightarrow \quad 0 \to Z^* \to Y^* \to X^* \text{ exact}$$

and similarly for $Z^* \to Y^* \to X^* \to 0$ exact. $\square$

$(-\otimes T)$

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(Skipped) □

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If $X \in \mathcal{C}$ has a left(right) dual, then $P \otimes X (X \otimes P)$ is projective.

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(2.10.8)

$$\Longrightarrow \mathbf{Hom}_{\mathcal{C}}(P \otimes X, Y) = \mathbf{Hom}_{\mathcal{C}}(P, Y \otimes X^*) \text{ exact.}$$

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□

## Corollary

*Let $\mathcal{C}$ be a multiring category with left duals.*
*$\mathbf{1} \in \mathcal{C}$ is projective if and only if $\mathcal{C}$ is semisimple.*

$1$ proj $\Rightarrow$ $X \cong 1 \otimes X$ proj $\forall X$ $\Rightarrow$ $\mathcal{C}$ semisimple

# 4.3. Semisimplicty of the unit object

Let $\mathcal{C}$ be a multiring category.
$\mathbf{End}_{\mathcal{C}}(\mathbf{1})$ is a semisimple algebra, i.e., we have

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Proof.
$\mathbf{End}_{\mathcal{C}}(\mathbf{1})$ is a commutative algebra. So we just need to prove:
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$$J \otimes J = \mathbf{Im}(a \otimes a) = \mathbf{Im}(a^2 \otimes \mathbf{1}) = 0$$
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$$J \otimes J = \mathbf{Im}(a \otimes a) = \mathbf{Im}(a^2 \otimes \mathbf{1}) = 0$$

$$K \otimes J = \mathbf{Im}_{K \otimes \mathbf{1}}(\mathbf{1} \otimes a) = \mathbf{Im}_{K \otimes \mathbf{1}}(a \otimes \mathbf{1}) = 0$$

$0 \to K \to \mathbf{1} \to J \to 0$ exact

$(- \otimes J)$

$\implies$ $0 \to 0 \to J \to 0 \to 0$ exact

$\implies$ $J = 0 \implies a = 0$.

□

# Component subcategories

Let $\{p_i\}_{i \in I}$ be the primitve idempotents of $\mathbf{End}(\mathbf{1})$ and $\mathbf{1}_i$ the image of $p_i$, then

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## Definition

*Let $\mathcal{C}$ be a multiring category. The* **component subcategories** $\mathcal{C}_{ij}$ *are defined as*

$$\mathcal{C}_{ij} := \mathbf{1}_i \otimes \mathcal{C} \otimes \mathbf{1}_j.$$

# Exercises

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- *The unit object $\mathbf{1}$ is simple.*
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## Corollary

*If $\mathcal{C}$ is a multiring category with left duals, then $\mathbf{1}$ is semisimple and*

$$\mathbf{1} = \bigoplus_{i \in I} \mathbf{1}_i,$$

*with $\mathbf{1}_i$ pairwise non-isomorphic simple objects.*

# Unit object $\mathbf{1}$ is simple in ring category with left duals

## Proof

Suppose $X$ is a simple subobject of $\mathbf{1}$.

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$$\implies \quad 0 \to X \to \mathbf{1} \to Y \to 0 \text{ exact}$$

$$\implies \quad 0 \to Y^* \to \mathbf{1} \to X^* \to 0 \text{ exact}$$

$$\implies \quad 0 \to X \otimes Y^* \to X \to X \otimes X^* \to 0 \text{ exact}$$

$$\implies \quad X \otimes X^* \cong X$$

$$\implies \quad \mathbf{1} \to X \otimes X^* \to X \text{ surjective composition morphism}$$

$$\implies \quad \mathbf{1} \twoheadrightarrow X \hookrightarrow \mathbf{1} \text{ nonzero composition morphism}$$

$$\implies \quad X = \mathbf{1}$$

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Let $\mathcal{C}$ be a ring category with left duals and $X \in \mathcal{C}_{ij}$, $Y \in \mathcal{C}_{jk}$ nonzero objects.

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- $X \otimes Y \neq 0$.
- length$(X \otimes Y) \geq$ length$(X)$length$(Y)$.
- If $\mathcal{C}$ is a ring category, then invertible objects are simple.
- If $X \otimes X^* \cong \mathbf{1}$, then $X$ is invertible.

# 4.4. Absence of self-extensions of the unit object

**Theorem**

*Let $\mathcal{C}$ be a finite ring category over $\Bbbk$ with simple object $\mathbf{1}$. If $\mathrm{char}(\Bbbk) = 0$, then $\mathbf{Ext}^1(\mathbf{1}, \mathbf{1}) = 0$.*

# 4.4. Absence of self-extensions of the unit object

### Theorem

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### Corollary

*Let $\mathcal{C}$ be a finite ring category over $\Bbbk$ with* **unique** *simple object $\mathbf{1}$.*
*If $\mathrm{char}(\Bbbk) = 0$, then $\mathcal{C}$ is equivalent to $\mathbf{Vec}$.*

# If $\mathrm{char}(\Bbbk) = 0$, then $\mathbf{Ext}^1(\mathbf{1}, \mathbf{1}) = 0$

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Assume there exists a $V$ such that

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Let $P$ be a projective cover of $\mathbf{1}$ and $E := \mathrm{Hom}(P, \mathbf{1})$.

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Let $P$ be a projective cover of $\mathbf{1}$ and $E := \mathrm{Hom}(P, \mathbf{1})$.

$\Longrightarrow$ $0 \subset \mathrm{Hom}(P, X) \subset \mathrm{Hom}(P, V)$, with
$\mathrm{Hom}(P, X) \cong E \cong \mathrm{Hom}(P, V)/\mathrm{Hom}(P, X)$

Let $\{v_0\}$ be a basis of $\mathrm{Hom}(P, X)$ and
let $\{v_0, v_1\}$ be a basis of $\mathrm{Hom}(P, V)$.

# If $\mathrm{char}(\Bbbk) = 0$, then $\mathbf{Ext}^1(\mathbf{1}, \mathbf{1}) = 0$

## Proof part 2

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Let $\chi_0 : A \to \Bbbk$ be the character defined by the action of $A$ on $E$.

An element $a \in A$ in the basis $\{v_0, v_1\}$ has the form

$$[a]_1 = \begin{pmatrix} \chi_0(a) & \chi_1(a) \\ 0 & \chi_0(a) \end{pmatrix}, \quad \chi_1 \in A^*$$

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Since $a \to [a]_1$ is a homomorphism we have

$$\chi_1(ab) = \chi_1(a)\chi_0(b) + \chi_0(a)\chi_1(b).$$

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$\implies$ $0 \subset X \otimes X \subset (X \otimes V) \oplus (V \otimes X) \subset V \otimes V$

$\implies$ $0 \subset \mathrm{Hom}(P, X \otimes X) \subset \mathrm{Hom}(P, (X \otimes V) \oplus (V \otimes X)) \subset \mathrm{Hom}(P, V \otimes V)$

$\implies$ $0 \subset \langle v_{00} \rangle \subset \langle v_{00}, v_{01}, v_{10} \rangle \subset \langle v_{00}, v_{01}, v_{10}, v_{11} \rangle$

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An element $a \in A$ in the basis $\{v_{00}, v_{01}, v_{10}, v_{11}\}$ has the form

$$[a]_2 = \begin{pmatrix} \chi_0(a) & \chi_1(a) & \chi_1(a) & \chi_2(a) \\ 0 & \chi_0(a) & 0 & \chi_1(a) \\ 0 & 0 & \chi_0(a) & \chi_1(a) \\ 0 & 0 & 0 & \chi_0(a) \end{pmatrix}, \quad \chi_2 \in A^*$$

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Since $a \to [a]_2$ is a homomorphism we have

$$\chi_2(ab) = \chi_0(a)\chi_2(b) + \chi_2(a)\chi_0(b) + 2\chi_1(a)\chi_1(b).$$

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For all $s \in \Bbbk$ we can define $\phi_s : A \to \Bbbk[t]$ by

$$\phi_s(a) = \sum_{m \geq 0} \chi_m(a) \frac{s^m t^m}{m!}$$

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$\phi_s$ is a family of pairwise distinct homomorphisms.

Contradiction: $A$ can have only finitely many $1$-dimensional representations over any extension field.

# 4.5. Grothendieck ring

Recall the Grothendieck group of a locally finite abelian category $\mathcal{C}$,

$$\mathbf{Gr}(\mathcal{C}) = \{[X] = \sum_{i \in I} [X : X_k] X_k : X_k \in \mathcal{C} \text{ simple}\}.$$

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Now let $\mathcal{C}$ be a multiring category over $\Bbbk$.

Definition

*The* **Grothendieck ring** *of* $\mathcal{C}$ *is the group* $\mathbf{Gr}(\mathcal{C})$ *with multiplication*

$$[X_i][X_j] := [X_i \otimes X_j] = \sum_{i \in I} [X_i \otimes X_j : X_k] X_k,$$

*called the* **fusion rule** *of* $\mathcal{C}$*.*

# 4.5. Frobenius-Perron dimension

Recall

**Definition**

*Let $A$ be a transitive unital $\mathbb{Z}_+$ ring with basis $I$.*
*The* **Frobenius-Perron dimension** $\mathbf{FPdim} : A \to \mathbb{C}$ *is defined for $X \in I$ as the maximal non-negative eigenvalue of the matrix of left multiplication by $X$ and extented to $A$ by additivity.*

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### Proposition

*Let $\mathcal{C}$ be a ring category with left duals.*
$\mathbf{Gr}(\mathcal{C})$ *is a transitive unital $\mathbb{Z}_+$ ring.*

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## Proposition

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In particular:

we can define the Frobenius-Perron dimension of objects.

# Quasi-tensor functor as unital ring hom. categorification

Suppose for $\mathcal{C}$, $\mathcal{D}$ multiring categories over $\mathbb{k}$
and $F : \mathcal{C} \to \mathcal{D}$ a quasi-tensor functor, i.e.,
$F$ is an exact and faithful $\mathbb{k}$-linear functor such that $F(\mathbf{1}) = \mathbf{1}$
with a functorial isomorphism $J : F(-) \otimes F(-) \to F(- \otimes -)$.

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*$F$ defines a homomorphism of unital $\mathbb{Z}_+$ rings*

$$[F] : \mathbf{Gr}(\mathcal{C}) \to \mathbf{Gr}(\mathcal{D}).$$

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## Proposition

*If $\mathcal{C}$ and $\mathcal{D}$ are tensor categories with finitely many classes of simple objects, then for all $X$ in $\mathcal{C}$ we have*

$$\mathbf{FPdim}_{\mathcal{D}}(F(X)) = \mathbf{FPdim}_{\mathcal{C}}(X).$$