$\mathbb{Z}_+$ - RINGS

MENU

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0. Motivation

Def A fusion cat is a finite semisimple k-lin Abelian rigid monoidal category for which the bifunctor $\otimes$ is linear on morphisms and $\mathrm{End}\,\mathbb{1} \cong k$

Monoidal: $\otimes, \mathbb{1}, \alpha, l, r$ $\quad$ k-linear: $\mathrm{Hom}(X,Y)$ is k-vect space

Abelian: $K \xrightarrow{k} \underbrace{X \xrightarrow{i} \mathrm{Im}(\varphi) \xrightarrow{j} Y}_{\varphi} \xrightarrow{c} C \quad \forall \varphi$

Semi-simple: $X = \bigoplus_{i=1} S_i$: S_i has exactly 2 subobj

rigid: $\forall X \ \exists\ {}^*X, X^*, ev_X, coev_X, ev'_X, coev'_X$

finite: • $\dim(\mathrm{Hom}(X,Y)) \in \mathbb{N}$ • $\forall X$ has finite length • $\forall X$ has proj cover
• $\mathrm{Obj}(\mathcal{C})/\cong$ is finite

Properties • Every fusion category has a weak fusion ring structure on it's Grothendieck group $Gr(\mathcal{C})$.

- Not every weak fusion ring has a fusion cat
- Fusion rings $\xrightarrow{\text{categorification}}$ fusion cats

  ↳ solving the pentagon equations.

1. Definitions and examples

Def (i) $\mathbb{Z}_+$ basis: Let A be a ring, free as $\mathbb{Z}$-module. basis of A, such that

$$b_i b_j = \sum_k c_{ij}^k b_k \qquad c_{ij}^k \in \mathbb{Z}^+.$$

(ii) A $\mathbb{Z}_+$ ring is a ring, free as a $\mathbb{Z}$ mod, with a fixed $\mathbb{Z}$ basis and $1 = \sum_{i \in I} c_i b_i$ $\quad c_i \in \mathbb{Z}_+$

(iii) Unital if $1 \in B$. $\{b_i\}$ els of B

B ← basis

Def I_0, τ

I_0: $1 = \sum_{i \in I_0} c_i b_i$, $\quad \tau(b_i) = \begin{cases} 1 & i \in I_0 \\ 0 & \text{otherwise} \end{cases}$

Def Based ring: $\mathbb{Z}_+$ ring for which $\exists \, * : I \to I$ and has an associated map $*: A \to A$ which is an anti involution

$$\underset{\left(\sum a_i b_i\right)^*}{a^*} = \sum a_i b_{i^*} \quad \& \quad \tau(b_i b_j) = \begin{cases} 1 & i = j^* \\ 0 & i \neq j^* \end{cases}$$

Def (Multi) Fusion ring: Multi fusion ring: based & finite, fusion ring: unital based finite

Examples

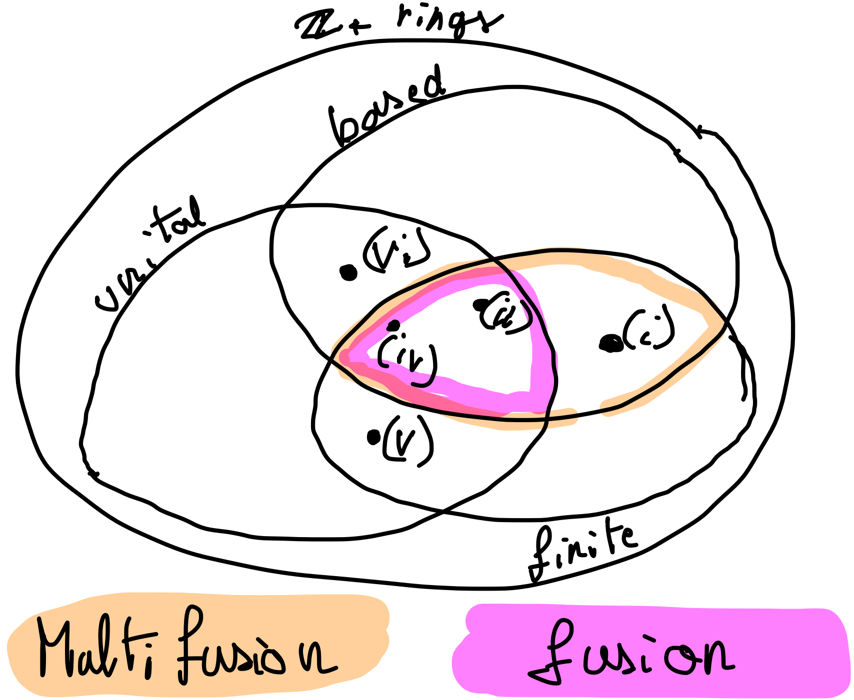
(i) $Mat_n(\mathbb{Z})$ with basis $\{E_{ij}\}$

(ii) Group rings

(iv) Ring of $\mathbb{C}$-reps of G

(v) " " $\mathbb{R}$-reps "

(vi) $\mathbb{C}$-Reps of compact Lie



2. Frobenius-Perron theorem

THEOREM 3.2.1. *(Frobenius-Perron) Let B be a square matrix with non-negative real entries.*

(1) *B has a non-negative real eigenvalue. The largest non-negative real eigenvalue $\lambda(B)$ of B dominates the absolute values of all other eigenvalues μ of B: $|\mu| \leq \lambda(B)$ (in other words, the spectral radius of B is an eigenvalue). Moreover, there is an eigenvector of B with non-negative entries and eigenvalue $\lambda(B)$.*

(2) *If B has strictly positive entries then $\lambda(B)$ is a simple positive eigenvalue, and the corresponding eigenvector can be normalized to have strictly positive entries. Moreover, $|\mu| < \lambda(B)$ for any other eigenvalue μ of B.*

(3) *If a matrix B with non-negative entries has an eigenvector $\mathbf{v}$ with strictly positive entries, then the corresponding eigenvalue is $\lambda(B)$.*

3. Frobenius-Perron dimensions

| Assume FR A with basis B

Def FP-dim: FPdim: $\forall X \in B$ $\quad \mathrm{FPdim}(X) = \lambda(\rho_{reg} \cdot X)$
$\lambda([C_X]^k_j)$.

Prop (1) $\mathrm{FPdim}: A \to \mathbb{C}$ is a ring homomorphism.

(2) Let $R := \sum_X \mathrm{FPdim}(X)\, X$

$$\boxed{\begin{aligned} yR &= \mathrm{FPdim}(y) R \\ Ry &= \mathrm{FPdim}(y) R \end{aligned}}$$

(3) FPdim is unique character that has vals ≥ 0 moreover they're > 0.

(4) $\mathrm{FPdim}(X^*) = \mathrm{FPdim}(X)$

$\Rightarrow \mathrm{FPdim}(X) = 1 \iff X X^* = X^* X = 1$
grouplike

Def Canonical regular element, FPdim (ring):
R as above is called ↑, $\mathrm{FPdim}(R) =: \mathrm{FPdim}(A)$.

Prop $f: A_1 \to A_2$ unital homomorphism whose matrix in the bases B_1, B_2 has entries ≥ 0 then

- $\mathrm{FPdim}(f(X)) = \mathrm{FPdim}(X)$.
- $\mathrm{FPdim}(A_2) f(R_1) = R_2 \mathrm{FPdim}(A_1)$

4. $\mathbb{Z}_+$-modules

Def (irreducible) $\mathbb{Z}_+$-module. Fusion ring A with basis $B = \{b_i\}$, then a $\mathbb{Z}_+$ module over A is an A module with fixed $\mathbb{Z}$ basis $\{m_i\}$ s.t.

$$b_i m_j = \sum \underbrace{a_{ij}^{\ell}}_{\in \mathbb{Z}_+} m_\ell$$

Prop Any fusion ring A has a finite amount of irreducible $\mathbb{Z}_+$-modules

5. Grading

Def. (faithful) grading, FPdim(A_g),

A faithful grading on a fusion ring A is a partition of the basis of A : $B = \coprod_{g \in G} B_g$ ← group

s.t. $b_g b_h = \sum_{b_k \in B_{gh}} c_{ij}^{k} b_k$.

$A = \bigoplus_{g \in G} A_g$, define $\boxed{\mathrm{FPdim}(A_g) = \mathrm{FPdim}(R_g)}$

Theorem $\mathrm{FPdim}(A_g) = \dfrac{\mathrm{FPdim}(A)}{|G|}$, if A faithfully graded.

Prop (Super fusion rings) $A_0 \subset A$ s.t. $\mathrm{Fdim}(A) = 2\,\mathrm{Fdim}(A_0)$

$\Rightarrow \exists\ \mathbb{Z}_2$ grading where A_0 is the identity component.

Def (Weakly) integral fusion ring. A fusion ring is weakly integral if $\mathrm{FPdim}(A) \in \mathbb{Z}_+$, and integral if $\mathrm{FPdim}(X) \in \mathbb{Z}_+ \ \forall X \in A$

Conjecture If fusion ring is weakly integral {reps} of associated braided fusion cat has finite image

Pop science Anyon models with weakly integral fusion can't be used for universal quantum computation.

6. Universal grading

Def Adjoint subring of a fusion ring A is the subring generated by $\{b_i b_i^*\}_{i\in I}$.

Prop Any one-sided A_{ad}-submodule of A is an A_{ad}-sub-bimodule

$\Rightarrow$ Can decompose any fusion ring A as

$$A = \bigoplus_{g\in G} \underbrace{A_g}_{\text{irred } A_{ad}\text{-bimodule}}$$

Moreover this decomposition is unique and

Prop Define $g \cdot h = k \iff \underset{\substack{\cap \\ A_g}}{x_g}\ \underset{\substack{\cap \\ A_h}}{x_h} \in A_k$

$A_L = A_{ad}$

then this is a well-defined operation that endows G with a group structure.

Def G is denoted as $U(A)$ and called the universal grading group of A.

Prop $U(A)$ is universal