Tensor Categories

Following: Tensor Categories by Etingof, Gelaki, Nikshych, Ostrik

Chapter 1 and half of Chapter 2

Tensor Categories

Overview of full seminar

Chapter 1	Homework and brief recap (1.1 to 1.9)
Chapter 2	2.1 to 2.10
Chapter 3	Skip unless needed
Chapter 4	4.1 to 4.9
Chapter 5	5.1 to 5.6, more if wanted
Chapter 6	6.1 to 6.3
Chapter 7	7.1 to 7.12
Chapter 8	8.1 to 8.14
Chapter 9	9.1 to 9.9 with 9.12

Abelian Cats
Monoidal Cats

Tensor Cats
Rep Cat of Hopf alg
Finite tensor Cats
Module Cats
Braided Cat
Fusion Cats

Overview

Chapter 1

1.1 Prerequisites
1.2 Additive Categories
1.3 Definition of abelian categories
1.4 Exact sequences
1.5 Length of objects and Jordan-Holder Th
1.6 Projective and injective objects
~~1.7 Higher Ext groups and group cohomology~~
1.8 Locally finite and finite abelian categories
1.9 Coalgebras
~~1.10 Coend construction~~
~~1.11 Deligne's tensor product~~
~~1.12 The finite dual~~
~~1.13 Pointed coalgebras and the coradical filtration~~

Chapter 2

2.1 Definition of a monoidal category
2.2 Basic properties of unit objects
2.3 First examples
2.4 Monoidal functors and their morphisms
2.5 Examples of monoidal functors

Prerequisites

1) Book assumes basic category theory
2) Disclaimer: Don't worry about ~~classes~~
3) Notation $X \in \mathcal{C}$ object

$\mathrm{Hom}_{\mathcal{C}}(X, Y)$ morphisms

$\phi : X \to Y$ or $X \xrightarrow{\phi} Y$

$\mathcal{C}^\vee$ - "dual" category

Summary of chapter 1: recalls theory of abelian categories [largely without proof]

k - field always (almost) algebraically closed.

Additive Categories

Definition - Additive category $\mathcal{C}$:

(A1): $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group. $(f+g)\circ h = f\circ h + g\circ h$

(A2): Zero object $0 \in \mathcal{C}$ s.t. $\mathrm{Hom}_{\mathcal{C}}(0,0) = 0$ $X \xrightarrow{0} Y$ $X \to 0 \to Y$

(A3): Direct sums: $X_1, X_2 \in \mathcal{C}$, there exists $Y \in \mathcal{C}$:

$$\begin{tikzcd}
X_1 \arrow[dr, "i_1"] & & X_1 \\
& Y \arrow[ur, "p_1"] \arrow[dr, "p_2"] & \\
X_2 \arrow[ur, "i_2"] & & X_2
\end{tikzcd}$$

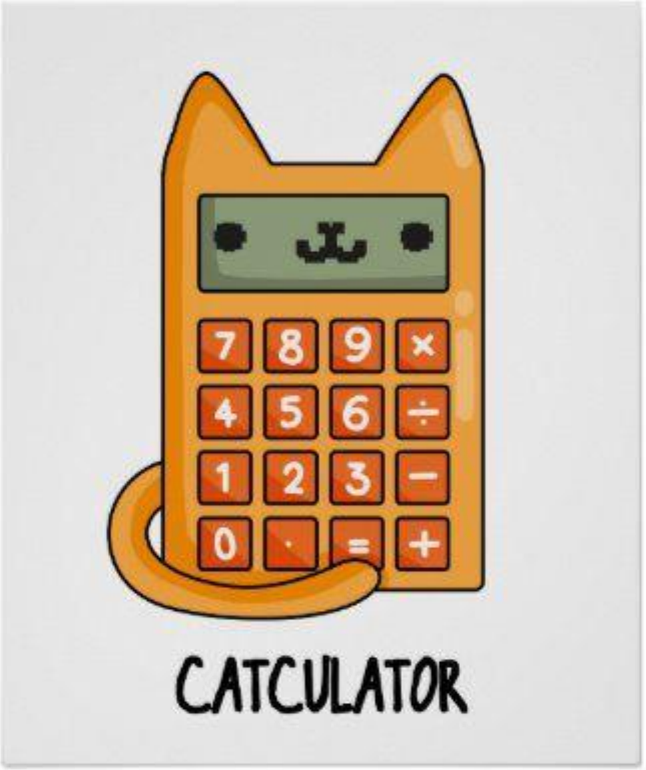
$$i_1p_1 + i_2p_2 = id_Y$$

Def - k-linear $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ k-vector space.

Def - additive functor $F: \mathcal{C} \to \mathcal{D}$ $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

a morphism of Ab. grps.

Additive Categories



Example: Ab - abelian groups.

$\psi, \phi \in \mathrm{Hom}_{Ab}(G, H)$

$(\psi + \phi)g = \phi(g) +_H \psi(g)$

Non-example: Sets

$f, g \in \mathrm{Hom}_{Set}(X, Y)$

Abelian categories

Def - Kernel of $f : X \to Y$.
(K, k)

$$\begin{tikzcd} & K' \arrow[dl, dotted] \arrow[d, "k'"] & \\ K \arrow[r, "k"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$fk = 0$

Def - Cokernel of $f : X \to Y$.
(C, c)

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "c"] \arrow[dr, "c'"'] & C \arrow[d, dotted] \\ & & C' \end{tikzcd}$$

$cf = 0$

Definition 1.3.1 - Abelian Category, additive.

For all $\phi : X \to Y$:

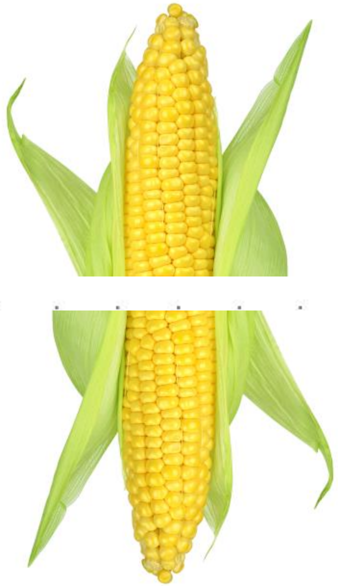
$$\left. K \xrightarrow{k} X \xrightarrow{i} \underline{I} \xrightarrow{j} Y \xrightarrow{c} C \right\}$$

(1) $ji = \phi$

(2) $(K, k) = \mathrm{Ker}\,\phi$ $\qquad$ $(C, c) = \mathrm{Coker}(\phi)$

(3) $(I, i) = \mathrm{Cok}(k)$ $\qquad$ $(I, j) = \mathrm{Ker}(c)$

$I = \mathrm{im}(\phi)$



Abelian categories

Example: left R-modules

ring with unit.

Non-example: torsion free ab. groups

$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ $\qquad$ $\mathbb{Z}/2\mathbb{Z}$ - not torsion free grp

Abelian categories

Def Monomorphism: f $\quad \mathrm{Ker}(f) = 0$

Def Epimorphism: f $\quad \mathrm{Coker}(f) = 0$

Def Subobject: $X \overset{i}{\hookrightarrow} Y$ or $X \subset Y$; mono. $\mathrm{Ker}(i) = 0$

Def Quotient: $Y \overset{\pi}{\twoheadrightarrow} Z$ $\quad \mathrm{Coker}(\pi) = 0$

Def Simple: has no non trivial subobj.
object.

Def Indecomposable $\quad X \neq X_1 \oplus X_2 \leftarrow \neq 0$
object

Abelian categories

Theorem 1.3.8 Mitchell - Freyd embedding.

Every abelian category is equivalent, as an additive category to:

full subcategory of left-modules of A. associative unital ring.

Warning A is not unique

Exact sequences

$$\cdots \dashrightarrow X_{i-1} \xrightarrow{f_{i-1}} \underset{\sim}{X_i} \xrightarrow{f_i} X_{i+1} \dashrightarrow \cdots$$

Exact at i^{th} position if $\mathrm{im}(f_{i-1}) = \ker(f_i)$.

Short exact sequences

$$0 \to X \to Z \to Y \to 0$$

"Extension of Y by X"

$\mathcal{C}$

X is a subobject of Z

$Y = Z/X$ is a quotient

$\mathrm{Ext}^1(Y, X)$

Equivalent SES: $\exists f$

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\wr"] & Z \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & X \arrow[r] & Z' \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}$$

$\cancel{\mathrm{Ext}^1(X, Y)} = \mathrm{SES} / \sim_{eq.}$

Length of objects and the Jordan-Holder theorem

Def $X \in \mathcal{C}$ is simple if no non-triv subobj.

Def $X \in \mathcal{C}$ is semisimple if direct sum of simple.

Schur's Lemma $X, Y \in C$ simple then $Hom_C(X, Y) = 0$ or isomorphism

Def Finite length:

$$0 = X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X_n = X$$

$$\underbrace{X_i / X_{i-1} \text{ simple}}$$

Length(X) = $\underline{\underline{n}}$

Theorem Jordan-Holder theorem:

Length(X) is well defined [if it exists]

Length of objects and the Jordan-Holder theorem

~~In an~~ abelian category every object has finite length. → Not true

Def 1.5.5

$\underbrace{\mathbb{Z}^{\oplus \mathbb{R}}}$

Theorem (Krull-Schmidt) Every $X \in \mathcal{C}$:

$$X = \bigoplus X_i \quad \text{when } \underset{=}{X_i} \text{ - indecomposable}$$

Multiplicity: for X simple $Y \in \mathcal{C}$ $\quad$ [Y,X]= multiplicity of X in any SH filtration of Y

Definition Grothendieck group $\quad Gr(\mathcal{C})$ - free abelian group gen by $\underset{=}{X_i}$ $\quad$ iso class of simples $\quad$ $\mathcal{O}(\mathcal{C})$

$$[Y] = \sum_i [Y : X_i] X_i$$

Projective and injective objects

Functor: $F : \mathcal{C} \to \mathcal{D}$

SES $0 \to X \to Z \to Y \to 0$

Left exact if $\cancel{0} \to F(X) \to F(Z) \to F(Y) \to 0$ is exact # right exact.

Example: $\underline{\mathbb{Z}} \in Ab$.

$0 \to G \to H \to K \to 0$

$0 \to Hom(\mathbb{Z}, G) \to Hom(\mathbb{Z}, H) \to Hom(\mathbb{Z}, K) \to 0$

Definition $P \in \mathcal{C}$ is projective if $\mathrm{Hom}_{\mathcal{C}}(\mathrm{P}, -)$ is exact.

Definition $I \in \mathcal{C}$ is injective if $\mathrm{Hom}_{\mathcal{C}}(-, \mathrm{I})$ is exact.

$\mathcal{C} \to \mathcal{D}$

$\mathcal{C} \to \mathcal{D}^{\vee} \leftarrow$

Projective cover $P(X)$ of $X \in \mathcal{C}$

$$P \to P(X) \twoheadrightarrow X$$

Injective hull $Q(X)$ for $X \in \mathcal{X}$

$$X \to Q(X)$$
$$\searrow \tilde{Q}$$

Locally finite and finite abelian categories

Definition Locally finite category $\mathcal{C}$:

$\mathcal{C}$ is ~~abelian,~~ $\mathbf{k}$-linear, and:

(i) $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ finite dim.

(ii) every $X \in \mathcal{C}$ has finite length.

~~emphasis~~

Def We denote the set of isomorphism classes of objects in X by $\mathcal{O}(\mathcal{C})$.

Locally finite and finite abelian categories

Definition Finite category $\mathcal{C}$: $\mathbb{k}$-linear

if equiv. to

Category fin dim A-mod.

A - fin dim $\mathbb{k}$-alg.

Definition Finite category $\mathcal{C}$:

$\mathcal{C}$ is abelian, **k**-linear, and:

loc. fin

(i) $\dim \mathrm{Hom}_{\mathcal{C}}(X, Y) < \infty$

(ii) $\mathrm{length}(X) < \infty$

(iii) Enough proj ~~modules~~ objects — every simple object has a proj. cover.

(iv) $|\mathcal{O}(\mathcal{C})| < \infty$.

Coalgebras

Definition - (Coalgebra) C - k vectorspace with comultiplication $\Delta : C \to C \otimes C$ and counit: $\epsilon : C \to \mathbf{k}$ s.t:

(i) Coassociativity $\quad (\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta$

(ii) Counit axiom $\quad (\epsilon \otimes Id) \circ \Delta = Id = (Id \otimes \epsilon) \circ \Delta$

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\Delta \otimes Id"] \\ C \otimes C \arrow[r, "Id \otimes \Delta"] & (C \otimes C) \otimes C \quad C \otimes (C \otimes C) \end{tikzcd}$$

Definition left co-module M $\quad$ v.sp M, $\quad \pi : M \to C \otimes M$

$$(\pi \otimes Id)(\pi(m)) = (Id \otimes \Delta)(\pi(m)) \qquad (Id \otimes \epsilon)(\pi(m)) = m$$

Chapter 2: Monoidal Categories

Definition of a Monoidal category

$1 \in \mathcal{C}$

Definition - Monoidal category $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ $\qquad \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \qquad \underline{\iota : 1 \otimes 1 \xrightarrow{\sim} 1}$$

Pentagram axiom

$$\begin{tikzcd}
 & ((W\otimes X)\otimes Y)\otimes Z \arrow[dl, "a_{W,X,Y}\otimes Id_Z"'] \arrow[dr, "a_{W\otimes X,Y,Z}"] & \\
(W\otimes(X\otimes Y))\otimes Z \arrow[d, "a_{W,X\otimes Y,Z}"] & \text{Commutes} & (W\otimes X)\otimes(Y\otimes Z) \arrow[d, "a_{W,X,Y\otimes Z}"] \\
W\otimes((X\otimes Y)\otimes Z) \arrow[rr, "Id_W\otimes a_{X,Y,Z}"] & & W\otimes(X\otimes(Y\otimes Z))
\end{tikzcd}$$

Unit axiom

$$\left.\begin{array}{l} L_1 : X \to 1 \otimes X \\ R_1 : X \to X \otimes 1 \end{array}\right\} \text{auto equivalence of } \mathcal{C}.$$

Definition of a Monoidal category

mon'l.
Subcategory: $D \subseteq C$

closed under $\otimes$

Opposite monoidal category

C^{op} $X \otimes^{op} Y = Y \otimes X$ monoidal cat

$a^{op} = a^{-1}$

$C^{op} \neq C^{\vee}$

Basic properties of unit objects

Alternative definition of monoidal category

Historic

Definition (Alt) - monoidal category $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$:

$r_X : X \otimes 1 \xrightarrow{\sim} X$

$l_X : 1 \otimes X \xrightarrow{\sim} X$

Pentagon axiom

as above

Triangle axiom

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (1 \otimes Y) \arrow[dl, "\mathrm{id} \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

First examples

Sets: Cat. of sets Unit one object.

$\otimes \to$ direct product.

Additive categories: fake example!

$\otimes = \oplus$

k-Vec - Vector spaces over k V, W v.sp

$V \otimes_k W$

k-vec $\oplus$

k-vec $\otimes$

First examples

R-modules - for R commutative unital ring.

generalisation of k-Vec.

Representations of G $\quad V, W \quad \rho_V, \rho_W : G \to End(V)$ or $End(W)$

$$G\text{-mod} \times G\text{-mod} \to Vec$$
$$V, \; W \qquad \underbrace{V \otimes W}$$

$$\rho_{V\otimes W} : G \to End(V \otimes W)$$
$$g \mapsto \rho_V(g) \otimes \rho_W(g)$$

$\Delta(g) = g \otimes g$

$[\rho_V \otimes 1 + 1 \otimes \rho_V \quad \mathfrak{g}\text{-LA}]$

First examples

The category $\mathcal{C}_G(A)$ for a group G, abelian group A.

obj: $\delta_g \longleftrightarrow g \in G$. $\qquad \delta_g \otimes \delta_h = \delta_{gh}$

mor: $\mathrm{Hom}(\delta_g, \delta_h) = A$ if $g = h$, 0 o.w.

The associativity isn't always straightforward:

$$\omega : G \times G \times G \to A \text{ is a 3-cocycle}$$

$$\omega(g_1 g_2, g_3, g_4)\omega(g_1, g_2, g_3 g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2 g_3, g_4)\omega(g_2, g_3, g_4) \quad \mathcal{C}_G^\omega$$

define: $a^\omega_{g,h,m} : (\delta_g \otimes \delta_h) \otimes \delta_m \longrightarrow \delta_g \otimes (\delta_h \otimes \delta_m)$

$\omega(g,h,m)\,\mathrm{Id}_{\delta_{ghm}}$

Functors of monoidal categories and their morphisms

Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ and $(\mathcal{C}', \otimes', \mathbf{1}', a', \iota')$ be two categories.

funct $C \to C'$

Definition - A monoidal functor is a pair (F, J) such that:

$$J_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$$

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "a'"] \arrow[d, "J_{X,Y} \otimes \mathrm{Id}"] & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "\mathrm{Id} \otimes J_{Y,Z}"] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "J_{X \otimes Y, Z}"] \arrow[r, phantom, "\text{Comm.}"] & F(X) \otimes (F(Y \otimes Z)) \arrow[d, "J_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(a)"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Functors of monoidal categories and their morphisms

Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ and $(\mathcal{C}', \otimes', \mathbf{1}', a', \iota')$ be two categories.

$\varphi : 1' \xrightarrow{\sim} F(1)$

Definition[Traditional] - A monoidal functor is a pair (F, J, φ) such that:

i) J satisfies above

ii)

$$\begin{tikzcd}
1' \otimes F(X) \arrow[r, "l_{F(X)}"] \arrow[d, "\varphi \otimes \mathrm{id}"'] \arrow[dr, phantom, "\circlearrowright"] & F(X) \arrow[d, "F(l_X^{-1})"] \\
F(1) \otimes F(X) \arrow[r, "J_{1,X}"] & F(1 \otimes X)
\end{tikzcd}$$

Monoidal functors are categories

Let (F_1, J_1) and (F_2, J_2) be monoiodal functors from $\mathcal{C}$ to $\mathcal{C}'$ as above:

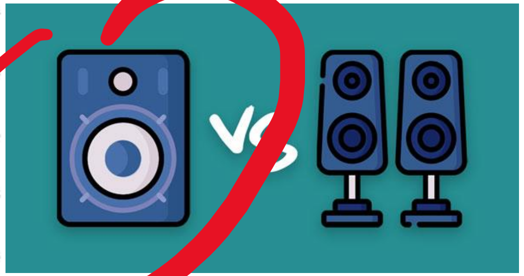
natural trans of F_1, F_2

A morphism $\eta : F_1 \to F_2$ such that:

$\eta_1 : \mathbf{1} \to \mathbf{1}$ is an iso

$$\begin{tikzcd}
F_1(X) \otimes F_1(Y) \arrow[r, "J_1"] \arrow[d, "\eta_X \otimes \eta_Y"] & F_1(X \otimes Y) \arrow[d, "\eta_{X \otimes Y}"] \\
F_2(X) \otimes F_2(Y) \arrow[r, "J_2"] & F_2(X \otimes Y)
\end{tikzcd}$$

Example of monoidal functors



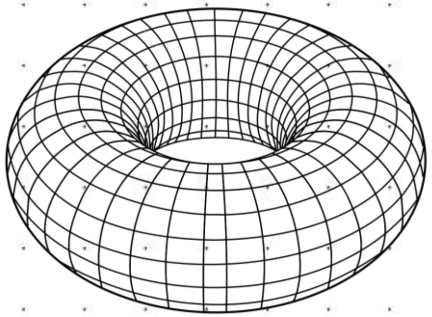
Forgetful functors:

$$\mathbf{Rep}(G) \to \mathbf{Vec}$$

$$\mathbf{Rep}(G) \to \mathbf{Rep}(H)$$ H subgroup G.

$(V, \rho) \in Rep(G)$ $(V, \rho|_H) \in Rep(H)$

Between graded spaces: $f : H \to G$



H-graded vector space.

$f_* : Vec_H \to Vec_G$

$V = \oplus V_h$ $V = \oplus V_{f(h)} \leftarrow \in G.$

Monoidal functors between categories of graded vector spaces

$\mathcal{C}_i = \mathcal{C}_{G_i}^{\omega_i}$ for $i = 1, 2$

$\mathrm{Hom}(\delta_g, \delta_h) = A$ if $g = h$.

Monoidal functor $F : \mathcal{C}_1 \to \mathcal{C}_2$

$$f : G_1 \to G_2$$

$J_{g,h} : F(\delta_g) \otimes F(\delta_h) \cong F(\delta_{gh})$

Monoidal structure on F

$\nu(g,h)\,\mathrm{Id}_{f(gh)}$

$\nu : G_1 \times G_1 \to A$

Where ω_1 and $f^*\omega_2$ are cohomologous in $Z^3(G_1, A)$.

$\omega_1 = f^*\omega_2 \cdot d_3(\nu)$

Converse:

any $f : G_1 \to G_2$, $\nu \in Z^2(G_1, A)$

$\omega_1 = f^*\omega_2\, d_3(\nu)$

Group actions on categories and equivariantization

$\mathrm{Aut}(\mathcal{C})$ obj: auto-equiv. of C
morph iso. of functor.

$\mathrm{Aut}_{\otimes}(\mathcal{C})$ mono auto equiv.

$\mathrm{Cat}(\mathrm{G}) = C_G(1)$

Definition - groups action of G on $\mathcal{C}$:

monoidal functor $T : Cat(G) \longrightarrow Aut(C)$

$\gamma_{g,h} : T_g \circ T_h = T_{g,h}$

$G \overset{\text{monoidally}}{\circlearrowleft} C$ $\quad T : Cat(G) \longrightarrow Aut_{\otimes}(C)$

Group actions on categories and equivariantization

Definition - G-equivariant object $(X^{\in \mathcal{C}}, u)$ $u = \{u_g : T_g(X) \xrightarrow{\sim} X\}$

$$\begin{tikzcd} T_g(T_h(X)) \arrow[r, "T_g(u_h)"] \arrow[d, "\gamma_{g,h}"'] & T_g(X) \arrow[d, "u_g"] \\ T_{gh}(X) \arrow[r, "u_{gh}"] & X \end{tikzcd}$$

Cat of G- equivariant objects of $\mathcal{C}$, $\mathcal{C}^G$: all G-equivariant objects in $\mathcal{C}$

There exists a forgetful functor: $\mathcal{C}^G \to \mathcal{C}$

The Mac Lane strictness theorem

Definition A monoidal category is strict if:

$C, \otimes, a, 1, \ell$

$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ $\quad X \otimes 1 = X = 1 \otimes X$

~~**example**~~ $\overline{Sets}$

Non-example: **Vec** or Vec_G^{ω}

$V \otimes k$

The Mac Lane strictness theorem

Theorem The Mac Lane strictness theorem

Any monoidal cat. is mono. equiv. to a strict cat.

warning Equiv. not isomorphism.

The Mac Lane strictness theorem

Definition Skeletal category

$$\mathrm{Ob}(\mathcal{C}) = \mathcal{C}$$

warning

$$\mathcal{C} \not\simeq \text{strict + skeletal}$$

The coherence theorem

$$
\begin{tikzcd}
(((X_1 \otimes X_2) \otimes \dots) \otimes X_n) \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "f=g"] & & (X_1 \otimes (X_2 \otimes \dots) \otimes) X_n)
\end{tikzcd}
$$

Rigid monoidal categories

Definition left dual object in a monoidal category $\mathcal{C}$. $X^* \to X \in \mathcal{C}$

$\exists \quad ev_X : X^* \otimes X \to 1 \qquad coev_X : 1 \to X \otimes X^*$

$$X \xrightarrow[\mathrm{Id}\otimes coev]{coev \otimes \mathrm{Id}} (X \otimes X^*) \otimes X \xrightarrow{a} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{Id} \otimes ev} X$$

Definition Rigid category   Category when every object is a left + right dual.

Rigid monoidal categories

Example Vec

Example Rep(G)