Translation functors

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Category $\mathcal{O}$ is decomposable:

$$\mathcal{O} = \bigoplus_{\chi} \mathcal{O}_\chi$$

Translation functor T_λ^μ

$$M \mapsto \mathrm{pr}_\mu(L(\bar{\nu}) \otimes \mathrm{pr}_\lambda(M))$$

for μ, λ compatible.

Overview

- Basic properties of T_λ^μ (7.1-7.2)
- Weyl group geometry
  - Facets (7.3)
  - Non-integral weights (7.4)
  - Key-lemma (7.5)
- Easy: translation from facet to facet closure
  - Verma modules (7.6)
  - Simple modules (7.7)
  - Categorical equivalences (7.8)
  - Simple modules again (7.9)
  - Characters (7.10)
- Hard: translations from facet closure to facet
  - Projective modules (7.11)
  - Translation from a wall (7.12-7.14)
  - Translation across a wall (7.15)
  - Self-dual projectives (7.16)

Facets

Definition

a facet F is a nonempty subset of E determined by a partition of ϕ^+ into disjoint subsets $\phi_F^0, \phi_F^+, \phi_F^-$:

$$\lambda \in F \Leftrightarrow \begin{cases} \langle \lambda + \rho, \alpha^\vee \rangle = 0 & \text{when } \alpha \in \phi_F^0, \\ \langle \lambda + \rho, \alpha^\vee \rangle > 0 & \text{when } \alpha \in \phi_F^+, \\ \langle \lambda + \rho, \alpha^\vee \rangle < 0 & \text{when } \alpha \in \phi_F^-, \end{cases}$$

Clearly the closure $\overline{F}$ is obtained by replacing $>$ by $\geq$ and $<$ by $\leq$.

Upper closure

Definition

The upper closure $\widehat{F}$ of the facet F is defined by the conditions

$$\lambda \in \widehat{F} \Leftrightarrow \begin{cases} \langle \lambda + \rho, \alpha^\vee \rangle = 0 & \text{when } \alpha \in \phi_F^0, \\ \langle \lambda + \rho, \alpha^\vee \rangle > 0 & \text{when } \alpha \in \phi_F^+, \\ \langle \lambda + \rho, \alpha^\vee \rangle \leq 0 & \text{when } \alpha \in \phi_F^-, \end{cases}$$

Theorem

Upper closures of facets have the following properties:

- *Each facet lies in the upper closure of a unique chamber.*
- *If the facet F lies in the upper closure of the chamber C, then $\overline{F} \subset \overline{C}$.*

Proposition

Let $\lambda \in \mathfrak{h}^*$, with stabilizer W_λ° in $W_{[\lambda]}$

- *For all $w \in W_{[\lambda]}$, we have $\lambda - w \cdot \lambda = \lambda^\sharp - w \cdot \lambda^\sharp$, while $(w \cdot \lambda)^\sharp = w \cdot \lambda^\sharp$.*
- *The stabilizer of $\lambda^\sharp$ in $W_{[\lambda]}$ is W_λ°.*
- *Suppose $\phi_{[\lambda]} = \phi_{[\mu]} = \phi_{[\lambda+\mu]}$. Then $(\lambda + \mu)^\sharp = \lambda^\sharp + \mu^\sharp$.*

Exercise

Show that W_λ° is the group which fixes pointwise the facet $F \subset E(\lambda)$ to which $\lambda^\sharp$ belongs; in turn, W_λ° is the group fixing $\overline{F}$ pointwise.

Verma modules

Theorem

Let $\lambda, \mu \in \mathfrak{h}^*$ be antidominant and compatible. Assume that $\lambda^\sharp$ lies in a facet F of $E(\lambda)$ relative to the dot-action of $W_{[\lambda]}$, while $\mu^\sharp$ lies in $\overline{F}$. Then $T_\lambda^\mu(M(w \cdot \lambda)) \cong M(w \cdot \mu)$ for all $w \in W_{[\lambda]}$. Similarly for the dual.

Translation to upper closures: simples again

Theorem

Let $\lambda, \mu \in \mathfrak{h}^*$ be antidominant and compatible. Assume that $\lambda^\sharp$ lies in a facet F of $E(\lambda)$, while $\mu^\sharp$ lies in $\overline{F}$.

1. *For all $w \in W_{[\lambda]}$, if $w \cdot \mu^\sharp$ lies in the upper closure of $w \cdot F$, then $T_\lambda^\mu L(w \cdot \lambda) \cong L(w \cdot \mu)$.*
2. *If $w \in W_{[\lambda]}$ but $w \cdot \mu^\sharp \notin w \cdot \widehat{F}$, then $T_\lambda^\mu L(w \cdot \lambda) = 0$.*

Projective modules

Theorem

Let $\lambda, \mu \in \mathfrak{h}^*$ be antidominant and compatible. Further assume that $\mu^\sharp$ lies in the closure of the facet F to which $\lambda^\sharp$ belongs. If $w \cdot \mu^\sharp$ lies in $\widehat{w \cdot F}$ for some $w \in W_{[\lambda]}$, then $T_\mu^\lambda(P(w \cdot \mu)) \cong P(w \cdot \lambda)$.

We will use Theorem 3.9(b): When a projective module $P \in \mathcal{O}$ is written as a direct sum of indecomposables, the number of those isomorphic to $P(\lambda)$ is equal to $\dim \operatorname{Hom}_{\mathcal{O}}(P, L(\lambda))$.

$P = T_\mu^\lambda(P(w \cdot \mu))$

$\operatorname{Hom}(P, L(w' \cdot \lambda)) \cong \operatorname{Hom}(P(w \cdot \mu), T_\lambda^\mu L(w' \cdot \lambda))$

$\neq 0$ only if

$T_\lambda^\mu L(w' \cdot \lambda) = L(w \cdot \mu)$

$w' \cdot \mu^\# \in \widehat{w' \cdot F}$

$\Rightarrow w' = w$

$\exists\, w' \neq w$, $w' \cdot \mu = w \cdot \mu \in \widehat{w \cdot F}$

$\Rightarrow$ $w' \cdot \lambda$, $w \cdot \lambda$ in the unique chamber $\widehat{C}$

$\Rightarrow w' \cdot \lambda = w \cdot \lambda$. $\square$

Translation From a Facet Closure

Theorem

Let $\lambda, \mu \in \mathfrak{h}^*$ be antidominant and compatible. Assume that $\mu^\sharp$ lies in the closure of the facet of $\lambda^\sharp$. Then for all $w \in W_{[\lambda]} = W_{[\mu]}$,,

$$\operatorname{ch} T_\mu^\lambda M(w \cdot \mu) = \operatorname{ch} \sum_{w' \in W_\mu^\circ / W_\lambda^\circ} M(ww' \cdot \lambda).$$

$(\mu \in H_\alpha)$

$T_\mu^\lambda M(ww' \cdot \lambda)$
$= M(ww' \cdot \mu)$
$= M(w \cdot \mu)$

In particular, all Verma modules which occur as quotients in a standard filtration of $T_\mu^\lambda(M(w \cdot \mu)$ have filtration multiplicity 1.

Use Theorem 3.7: If M has a standard filtration, then for all $\lambda \in \mathfrak{h}^*$ we have

$$(M : M(\lambda)) = \dim \operatorname{Hom}_{\mathcal{O}}(M, M(\lambda)^\vee).$$

$$\mathrm{Hom}_{\mathcal{O}}(T_\mu^\lambda(M(w\cdot\mu)),\ M(ww'\cdot\lambda)^\vee)$$

$$\cong \mathrm{Hom}_{\mathcal{O}}(M(w\cdot\mu),\ T_\lambda^\mu M(ww'\cdot\lambda)^\vee)$$

$$= \mathrm{Hom}_{\mathcal{O}}(M(w\cdot\mu),\ M(ww'\cdot\mu)^\vee)$$

if $w\cdot\mu = ww'\cdot\mu$

$w'\cdot\lambda = \lambda$

Translation From a Facet Closure

Corollary

Let λ, μ be as in the theorem. For arbitrary $M \in \mathcal{O}_{\mu}$ we have $\mathrm{ch}\, T_\lambda^\mu T_\mu^\lambda M = \underbrace{|W_\mu^\circ / W_\lambda^\circ|}\, \mathrm{ch} M$.

Example

Theorem (Theorem 4.10)

Let λ be antidominant and integral. Then a standard filtration of $P(\lambda)$ involves all of the distinct $M(w \cdot \lambda)$ exactly once. Thus $[M(w \cdot \lambda) : L(\lambda)] = 1$ for all $w \in W$.

Proposition

Let $\lambda, \mu \in \mathfrak{h}^*$ be antidominant and compatible, with $\mu^\sharp$ in the closure of the chamber containing $\lambda^\sharp$. Denote by w_λ the longest element in $W_{[\lambda]} = W_{[\mu]}$ and by w_μ° the longest element in W_μ°. Then $T_\mu^\lambda M(w_\lambda \cdot \mu) \cong P(w_\lambda w_\mu^\circ \cdot \lambda)$. Moreover, the Verma modules occurring as quotients in a standard filtration of $P(w_\lambda w_\mu^\circ \cdot \lambda)$ are the $M(\underline{w_\lambda w \cdot \lambda})$ with $w \in W_\mu^\circ$; each has multiplicity 1.

$\mu = -\rho$ $\quad\quad$ $w_\mu^\circ = W$

λ integral

Proof λ regular, $|W_\lambda^0| = 1$

$w_\lambda \cdot \mu$ is dominant $M(w_\lambda \cdot \mu)$ is projective.

$P = T_\mu^\lambda M(w_\lambda \cdot \mu)$ is projective

$\boxed{M(w_\lambda w \cdot \lambda) \text{ with } w \in W_\mu^0}$

- $w_\mu^0 \cdot \lambda \geq w \cdot \lambda$ for all $w \in W_\mu^0$

$$w_\lambda w_\mu^0 \cdot \lambda \leq w_\lambda w \cdot \lambda \qquad (3.6)$$

$M(w_\lambda w_\mu^0 \cdot \lambda)$ is a quotient of P.

$\Rightarrow$ $P(w_\lambda w_\mu^0 \cdot \lambda)$ is a summand of P.

- $M(w_\lambda w^0 \cdot \lambda) \hookrightarrow M(w_\lambda w \cdot \lambda)$

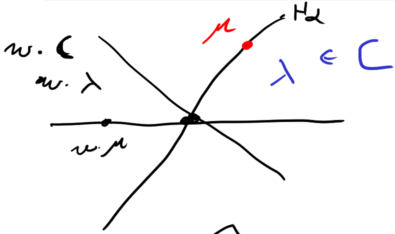
$0 \neq [M(w_\lambda w \cdot \lambda) : L(w_\lambda w^0 \cdot \lambda)] = (P(w_\lambda w_\mu^0 \cdot \lambda) : M(w_\lambda w \cdot \lambda))$ $\neq 0$

$\square$

Translation from a wall

Theorem

Let λ, μ be antidominant and compatible. Assume that λ is regular, so $\lambda^\sharp$ lies in a chamber C, while $\mu^\sharp$ lies in a single wall of C corresponding to $\alpha > 0$. Suppose w $\in W_{[\lambda]}$ satisfies $\boxed{w\alpha > 0,}$ so $\ell(ws) > \ell(w)$ with s $= s_\alpha$.



$w\alpha > 0 \iff \ell(ws) \geq \ell(w)$

$w\alpha \in \Phi^-$

$w \cdot C$

$\langle w \cdot \lambda + \rho, (w\alpha)^\vee \rangle$
$= \langle \lambda + \rho, \alpha^\vee \rangle < 0$

$\langle w \cdot \mu + \rho, (w\alpha)^\vee \rangle$
$= \langle \mu + \rho, \alpha^\vee \rangle = 0$

$w \cdot \mu \in \widehat{w \cdot C}$

$ws \cdot \lambda > w \cdot \lambda$

Translation from a wall

$$W_\mu^0 = \{ id , s_\alpha \}$$

$$T_\lambda^\mu L(w\cdot\lambda) = L(w\cdot\mu)$$

$$T_\lambda^\mu L(ws\cdot\lambda) = 0$$

$$[T_\lambda^\mu T_\mu^\lambda L(w\cdot\mu) : L(w\cdot\mu)] = 2 \quad [L(w\cdot\mu)]$$

$$T_\lambda^\mu \underbrace{T_\mu^\lambda L(w\cdot\mu)} = \underline{L(w\cdot\mu)} \oplus \underline{L(w\cdot\mu)} \oplus \underline{\cdots}$$

$$T_\mu^\lambda (M(w\cdot\mu)) : \quad \begin{array}{c} M(w\cdot\lambda) \\ | \\ M(ws\cdot\lambda) \end{array}$$

$$T_\mu^\lambda (L(w\cdot\mu)) = \quad \begin{array}{ccc} L(w\cdot\lambda) & & \\ | & & \\ ? & & L(ws\cdot\lambda) \\ & & L(w'\cdot\lambda) \\ | & & \\ L(w\cdot\lambda) & & \end{array}$$

$$\Rightarrow \quad T_\lambda^\mu (L(w'\cdot\lambda)) = 0, \quad w's\cdot\lambda < w'\cdot\lambda$$

Theorem

$ws \cdot \lambda > w \cdot \lambda$

1. *There is a short exact sequence*

$$0 \to \boxed{M(ws \cdot \lambda)} \to T_\mu^\lambda M(w \cdot \mu) \to M(w \cdot \lambda) \to 0.$$

2. $\operatorname{hd} T_\mu^\lambda M(w \cdot \mu) \cong L(w \cdot \lambda)$. *In particular,* $T_\mu^\lambda M(w \cdot \mu)$ *is indecomposable and the above exact sequence nonsplit.*
3. *The module* $T_\mu^\lambda L(w \cdot \mu)$ *is self-dual, with head and socle both isomorphic to* $L(w \cdot \lambda)$*; thus* $T_\mu^\lambda L(w \cdot \mu)$ *is indecomposable.*
4. $[T_\mu^\lambda L(w \cdot \mu) : L(w \cdot \lambda)] = 2$.
5. $[T_\mu^\lambda L(w \cdot \mu) : L(ws \cdot \lambda)] = 1$.
6. *Let* $w' \in W_{[\lambda]}$ *with* $w' \cdot \lambda \neq w \cdot \lambda$.
   If $[T_\mu^\lambda L(w \cdot \mu) : L(w' \cdot \lambda)] > 0$*, then* $w's \cdot \lambda < w' \cdot \lambda$.
   Moreover, $T_\lambda^\mu L(w' \cdot \lambda) = 0$.
7. *If* $w' \in W_{[\lambda]}$ *satisfies* $w's \cdot \lambda < w' \cdot \lambda$*, then*

$$\operatorname{Ext}_{\mathcal{O}}(L(w \cdot \lambda), L(w' \cdot \lambda)) \cong \operatorname{Hom}_{\mathcal{O}}(\operatorname{Rad} T_\mu^\lambda L(w \cdot \mu), L(w' \cdot \lambda)).$$

Proof]

1) $[T_\mu^\lambda M(w\cdot\mu)] = [M(w\cdot\lambda)] + [M(ws\cdot\lambda)]$

$ws\cdot\lambda > w\cdot\lambda \qquad (3.7)$

$M(ws\cdot\lambda)$ is a submodule

$$0 \to M(ws\cdot\lambda) \to T_\mu^\lambda(M(w\cdot\mu)) \to M(w\cdot\lambda) \to 0$$

2) $\mathrm{Hom}(T_\mu^\lambda(M(w\cdot\mu)), L(w'\cdot\lambda))$

$\cong \mathrm{Hom}(M(w\cdot\mu), T_\lambda^\mu(L(w'\cdot\lambda)))$

$\neq 0$ only if $w'\cdot\mu = w\cdot\mu$, $w'\cdot\mu \in w'\cdot\widehat{C}$

$\begin{cases} w' = w \\ w' \cancel{= ws} \end{cases} \Rightarrow \mathrm{Hd}(T_\mu^\lambda(M(w\cdot\mu))) = L(w\cdot\lambda)$

3) $L(w\cdot\mu)$ is self-dual.

$$T_\mu^\lambda(M(w\cdot\mu)) \twoheadrightarrow T_\mu^\lambda(L(w\cdot\mu))$$

$$\mathrm{Hd}(T_\mu^\lambda L(w\cdot\mu)) = L(w\cdot\lambda) = \mathrm{Soc}(T_\mu^\lambda(L(w\cdot\mu))$$

4) $[T_\mu^\lambda L(w\cdot\mu) : L(w\cdot\lambda)] \geq 2$

$$\hookrightarrow [T_\lambda^\mu T_\mu^\lambda L(w\cdot\mu) : T_\mu^\lambda L(w\cdot\lambda)] = 2$$

5) $[T_\mu^\lambda L(w\cdot\mu) : L(ws\cdot\lambda)] = 1$

$$M(ws\cdot\lambda) \hookrightarrow T_\mu^\lambda(M(w\cdot\mu)) \twoheadrightarrow T_\mu^\lambda(L(w\cdot\mu))$$

$L(w\cdot\lambda)$

↯

6] $[T_\mu^\lambda \, L(w\cdot\mu) : L(w'\cdot\lambda)] > 0$

$w' \neq w$

$\Rightarrow T_\lambda^\mu \, L(w'\cdot\lambda) = 0$

$w's\cdot\lambda < w'\cdot\lambda$

- $w' = ws$

- $w\cdot\lambda = wss\cdot\lambda < ws\cdot\lambda$ ✓

- $T_\lambda^\mu(L(ws\cdot\lambda)) = 0$ since $ws\cdot\mu$ is not upper closure of $ws\cdot\lambda$ ✓.

- $w' \neq w, ws \Rightarrow L(w'\cdot\mu) \neq L(w\cdot\mu)$

$[T_\lambda^\mu T_\mu^\lambda L(w\cdot\mu) : T_\lambda^\mu \, L(w'\cdot\lambda)] > 0$ if $T_\lambda^\mu L(w'\lambda)$ to

Thus $T_\lambda^\mu L(w'\cdot\lambda) = 0$ $\quad w' \eta \cdot \lambda < w' \cdot \lambda$

$\nexists$ $w' \eta \cdot \lambda < w' \cdot \lambda \Rightarrow \underline{T_\lambda^\mu L(w'\cdot\lambda)) = 0.}$

$$0 \to \mathrm{Rad}(T_\mu^\lambda L(w\cdot\mu)) \to T_\mu^\lambda L(w\cdot\mu) \to L(w\cdot\lambda) \to 0$$

$\downarrow \mathrm{Hom}(\cdot\ ,\ L(w'\cdot\lambda))$

$\to \cancel{\mathrm{Hom}(T_\mu^\lambda L(w\cdot\mu), L(w'\cdot\lambda))}^{\,0} \to \mathrm{Hom}(\mathrm{Rad}(\cdots, L(w'\cdot\lambda))$

$\to \mathrm{Ext}(L(w\cdot\lambda), L(w'\cdot\lambda)) \to \cancel{\mathrm{Ext}(T_\mu^\lambda(L(w\cdot\mu)), L(w'\cdot\lambda))}$

$\to \cdots$

$\square$.

Exercise

Keep the hypotheses of the theorem. For all $w' \in W_{[\lambda]}$, prove that

$$[T_\mu^\lambda L(w \cdot \mu) : L(w' \cdot \lambda)] \leq 2[M(ws \cdot \lambda) : L(w' \cdot \lambda)].$$

Wall-crossing functors

μ is contained in the s-wall

$$\theta_s = \theta_s^\lambda = T_\mu^\lambda T_\lambda^\mu$$

$$\theta_s L(w\cdot\lambda) = \begin{array}{c} L(w\cdot\lambda) \\ | \\ ? \\ | \\ L(w\cdot\lambda) \end{array}$$

$$M \xrightarrow{?} \theta_s(M) \to \mathrm{Sa}_s(M) \to 0$$

$$\mathrm{Hom}(M, T_\mu^\lambda T_\lambda^\mu(M)) \cong \mathrm{Hom}(T_\lambda^\mu(M), T_\lambda^\mu(M))$$

F

Self-dual projectives

Theorem

Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then:

- $P(\lambda) \cong P(\lambda)^\vee$.
- *Any standard filtration of $P(\lambda)$ involves each $M(w \cdot \lambda)$ with $w \in W_{[\lambda]}$ precisely once as a quotient. Equivalently, BGG Reciprocity implies that $[M(w \cdot \lambda) : L(\lambda)] = 1$ for all $w \in W_{[\lambda]}$.*

- λ antidominant + regular

$\Rightarrow \lambda_0 = w_\lambda \cdot \lambda$ is dominant $\quad M(\lambda_0) = P(\lambda_0)$

- $M(\lambda) = L(\lambda)$ and socle of each $M(w \cdot \lambda)$

- $w_\lambda = s_{\alpha_1} \cdots s_{\alpha_m}$

$\Theta = \theta_{s_1} \cdots \theta_{s_m}$

$L(\lambda) \hookrightarrow P(\lambda_0)$

self-dual $\quad \theta \downarrow \quad$ proj

$\theta(L(\lambda)) \hookrightarrow \boxed{\theta(P(\lambda_0))}$

$\theta_s(M(w\cdot\lambda)) = T_\mu^\lambda(M(w\cdot\mu))$

$\theta_s(L(w\cdot\lambda)) = \begin{array}{c} L(w\cdot\lambda) \\ | \\ \sim \\ | \\ L(w\cdot\lambda) \end{array}$

socle $L(\lambda)$

$\theta_s(M(w\cdot\lambda)) = \begin{array}{c} \boxed{M(w\cdot\lambda)} \\ | \\ M(ws\cdot\lambda) \end{array}$

$M(w\cdot\lambda) \to$

$\boxed{M(\lambda)}$

$M(ws\cdot\lambda) \hookrightarrow \theta_s(M(w\cdot\lambda)) \twoheadrightarrow M(w\cdot\lambda) \to 0$

$[\theta(P(\lambda_0)) : M(\lambda)] = 1$

$\theta(L(\lambda)) \hookrightarrow \theta(P(\lambda_0)) \twoheadrightarrow P(\lambda) \twoheadrightarrow L(\lambda)$

is non-zero

$\theta(L(\lambda)) \twoheadrightarrow P(\lambda)$

$L(\lambda)$

$$\underbrace{Q(\lambda)} \hookrightarrow O(L(\lambda)) \hookrightarrow O(P(\lambda))$$

$$P(\cdot\lambda)^\vee = Q(\lambda) \supseteq P(\lambda)$$

- antidominant, non-regular.

$\lambda = \mu - \rho_n$ is regular.

$$T_\lambda^\mu P(\lambda) = P(\mu)^{\oplus 2}$$

$\square$