Translation functors

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Category $\mathcal{O}$ is decomposable:

$$\mathcal{O} = \bigoplus_{\chi} \mathcal{O}_\chi$$

Translation functor T_λ^μ

$\nu = \mu - \lambda$

$$M \mapsto \mathrm{pr}_\mu(L(\bar{\nu}) \otimes \mathrm{pr}_\lambda(M))$$

for μ, λ compatible.

Overview

- Basic properties of T_λ^μ (7.1-7.2)
- Weyl group geometry
  - Facets (7.3)
  - Non-integral weights (7.4)
  - Key-lemma (7.5)
- Easy: translation from facet to facet closure
  - Verma modules (7.6)
  - Simple modules (7.7)
  - Categorical equivalences (7.8)
  - Simple modules again (7.9)
  - Characters (7.10)
- Hard: translations from facet closure to facet
  - Projective modules (7.11)
  - Translation from a wall (7.12-7.14)
  - Translation across a wall (7.15)
  - Self-dual projectives (7.16)

Facets

Definition

a facet F is a nonempty subset of E determined by a partition of ϕ^+ into disjoint subsets $\phi_F^0, \phi_F^+, \phi_F^-$:

$$\lambda \in F \Leftrightarrow \begin{cases} \langle \lambda + \rho, \alpha^\vee \rangle = 0 & \text{when } \alpha \in \phi_F^0, \\ \langle \lambda + \rho, \alpha^\vee \rangle > 0 & \text{when } \alpha \in \phi_F^+, \\ \langle \lambda + \rho, \alpha^\vee \rangle < 0 & \text{when } \alpha \in \phi_F^-, \end{cases}$$

Clearly the closure $\overline{F}$ is obtained by replacing $>$ by $\geq$ and $<$ by $\leq$.

Upper closure

Definition

The upper closure $\widehat{F}$ of the facet F is defined by the conditions

$$\lambda \in \widehat{F} \Leftrightarrow \begin{cases} \langle \lambda + \rho, \alpha^\vee \rangle = 0 & \text{when } \alpha \in \phi_F^0, \\ \langle \lambda + \rho, \alpha^\vee \rangle > 0 & \text{when } \alpha \in \phi_F^+, \\ \langle \lambda + \rho, \alpha^\vee \rangle \leq 0 & \text{when } \alpha \in \phi_F^-, \end{cases}$$

Theorem

Upper closures of facets have the following properties:

- *Each facet lies in the upper closure of a unique chamber.*
- *If the facet F lies in the upper closure of the chamber C, then $\overline{F} \subset \overline{C}$.*

Proposition

Let $\lambda \in h^*$, with stabilizer W°_λ in $\boxed{W_{[\lambda]}}$

- *For all $w \in W_{[\lambda]}$, we have $\lambda - w \cdot \lambda = \lambda^\sharp - w \cdot \lambda^\sharp$, while $(w \cdot \lambda)^\sharp = w \cdot \lambda^\sharp$.*
- *The stabilizer of $\lambda^\sharp$ in $W_{[\lambda]}$ is W°_λ.*
- *Suppose $\phi_{[\lambda]} = \phi_{[\mu]} = \phi_{[\lambda+\mu]}$. Then $(\lambda + \mu)^\sharp = \lambda^\sharp + \mu^\sharp$.*

Exercise

Show that W°_λ is the group which fixes pointwise the facet $F \subset E(\lambda)$ to which $\lambda^\sharp$ belongs; in turn, W°_λ is the group fixing $\overline{F}$ pointwise.

Key lemma

Lemma

Let $\lambda, \mu \in h^*$ be compatible, with $\nu := \mu - \lambda \in \Lambda$ and $\overline{\nu}$ the unique W-conjugate of ν lying in Λ^+ . Assume that $\lambda^\sharp$ lies in a facet F of $E(\lambda)$, while $\mu^\sharp$ lies in the closure $\overline{F}$. Then for all weights $\nu' \neq \nu$ of $L(\overline{\nu})$, the weight $\lambda + \nu'$ is not linked by $W_{[\lambda]} = W_{[\mu]}$ to $\lambda + \nu = \mu$.

Verma modules

Theorem

Let $\underline{\lambda, \mu \in}$ h^* be antidominant and compatible. Assume that $\lambda^\sharp$ lies in a facet F of $E(\lambda)$ relative to the dot-action of $W_{[\lambda]}$, while $\mu^\sharp$ lies in $\overline{F}$. Then $T_\lambda^\mu(M(\underline{w \cdot \lambda})) \cong M(w \cdot \mu)$ for all $w \in W_{[\lambda]}$. Similarly for the dual.

- $w \cdot \mu - w \cdot \lambda = w\mu - w\lambda \to$ some W-orbit

as $\nu = \mu - \lambda$

$w \cdot \lambda^\# \in w \cdot F$, $w \cdot \mu^\# \in \overline{w \cdot F}$

$\Rightarrow$ Jerg lemma: $w \cdot \lambda + \nu'$ is not linked to $w \cdot \mu$

$L(\overline{\nu}) \otimes M(w \cdot \lambda)$ has a filtration whose quotients are $M(w \cdot \lambda + \nu')$

So after projecting only $M(w \cdot \mu)$ survives.

$$T_\lambda^\mu (M(w \cdot \lambda)) = M(w \cdot \mu)$$

$\square$.

Simple modules

Proposition

Let $\lambda, \mu \in h^*$ be antidominant and compatible. Assume that $\lambda^\sharp$ lies in a facet F of $E(\lambda)$ relative to the dot-action of $W_{[\lambda]}$, while $\mu^\sharp$ lies in $\overline{F}$. If $w \in W_{[\lambda]}$, then $T_\lambda^\mu L(w \cdot \lambda)$ is either 0 or else isomorphic to $L(w \cdot \mu)$.

Use Theorem 3.3(c): Any nonzero homomorphism $M(\mu) \to M(\lambda)^\vee$ has the simple submodule $L(\lambda)$ as its image.

$\exists \quad M(w\cdot\lambda) \twoheadrightarrow L(w\cdot\lambda)$ combining

$T_\lambda^\mu \downarrow \quad M(w\cdot\mu) \twoheadrightarrow T_\lambda^\mu(L(w\cdot\lambda)) \qquad \Rightarrow T_\lambda^\mu(L(w\cdot\lambda))$

$\exists \quad L(w\cdot\lambda) \hookrightarrow M(w\cdot\lambda)^\vee \qquad \cong L(w\cdot\mu).$

$T_\lambda^\mu \downarrow \quad T_\lambda^\mu(L(w\cdot\lambda)) \hookrightarrow M(w\cdot\mu)^\vee$ if the map is non-zero. $\square$

Exercise

In the proposition, assume that $T_\lambda^\mu L(w \cdot \lambda) \cong L(w \cdot \mu)$. For arbitrary $M \in \mathcal{O}$, prove that $[M : L(w \cdot \lambda)] = [T_\lambda^\mu(M) : L(w \cdot \mu)]$.

Composition series of M

$$M = M_0 \supseteq M_1 \supset \cdots \quad M_r$$

$$M_i / M_{i+1} \cong L(\lambda_i)$$

apply exact functor. obtain composition series of

$$T_\lambda^\mu(M_0) \supseteq T_\lambda^\mu(M_1) \supseteq \cdots \supseteq T_\lambda^\mu(M_r)$$

$$T_\lambda^\mu(M_i) / T_\lambda^\mu(M_{i+1}) \cong \begin{cases} L(\mu_i) \\ 0 \end{cases}$$

$$[M : L(w\cdot\lambda)] \leq [T_\lambda^\mu(M) : L(w\cdot\mu)].$$

Example: $\mathfrak{sl}_2(\mathbb{C})$.

$\lambda \geq 0$ lies in $F = \mathbb{Z}^+$

$\mu = -1$

$$\begin{array}{ccccccccc} 0 & \to & M(-\lambda-2) & \to & M(\lambda) & \to & L(\lambda) & \to & 0 \\ & & \wr\! \| & & & & & & \\ T_\lambda^\mu \big\downarrow & & L(-\lambda-2) & & & & & & \end{array}$$

$$0 \to M(\mu) \to M(\mu) \to \underbrace{T_\lambda^\mu(L(\lambda))}_{} \to 0 \qquad T_\lambda^\mu(L(\lambda)) = 0$$

$$T_\lambda^\mu(L(-\lambda-2)) = M(\mu) = L(\mu)$$

Equivalence of categories

> ***Proposition***
>
> *Let $\lambda, \mu \in h^*$ be antidominant and compatible. Assume that $\lambda^\sharp$ and $\mu^\sharp$ lies in the same facet F of $E(\lambda) = E(\mu)$ relative to $W_{[\lambda]} = W_{[\mu]}$. Then T_λ^μ induces an isomorphism between the Grothendieck groups $K(\mathcal{O}_\lambda)$ and $K(\mathcal{O}_\mu)$ of the associated blocks, sending $[M(w \cdot \lambda)]$ to $[M(w \cdot \mu)]$ and $[L(w \cdot \lambda)]$ to $[L(w \cdot \mu)]$ for all $w \in W_{[\lambda]}$.*

- $[M(w \cdot \lambda)]$ give $\mathbb{Z}$-basis of $K(\mathcal{O}_\lambda)$.
- $T_\mu^\lambda T_\lambda^\mu (M(w \cdot \lambda)) = T_\mu^\lambda (M(w \cdot \mu)) = M(w \cdot \lambda)$
- $T_\mu^\lambda T_\lambda^\mu (L(w \cdot \lambda)) = \begin{cases} \cancel{0} \\ L(w \cdot \lambda) \end{cases}$

$\square$

Equivalence of categories

Theorem

Let $\lambda, \mu \in h^*$ be antidominant and compatible. Assume that $\lambda^\sharp$ and $\mu^\sharp$ lies in the same facet lie in the same facet for the action of $W_{[\lambda]} = W_{[\mu]}$ on $E(\lambda) = E(\mu)$, the functors T_λ^μ and T_μ^λ define inverse equivalences of categories between $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$. In particular, T_λ^μ sends simple modules to simple modules.

Use adjointness

$$\mathrm{Hom}_{\mathcal{O}}(T_\lambda^\mu(M), N) \cong \mathrm{Hom}_{\mathcal{O}}(M, T_\mu^\lambda(N))$$

$$T_\mu^\lambda T_\lambda^\mu \cong \mathrm{id}_{\mathcal{O}_\lambda}$$

$$\mathrm{Hom}_{\mathcal{O}}(T_\mu^\lambda T_\lambda^\mu(M), M) \cong \mathrm{Hom}_{\mathcal{O}}(T_\lambda^\mu(M), T_\lambda^\mu(M))$$

$$\varphi_M \qquad \longleftarrow \qquad \mathrm{id}_{T_\lambda^\mu(M)}$$

$$
\begin{tikzcd}
0 \arrow[r] & T^{\lambda}_{\mu} T^{\mu}_{\lambda}(M) \arrow[r] \arrow[d, "\mathrm{IH}"', "\wr\,\varphi_M"] \arrow[dr, phantom, "\circlearrowright"] & T^{\lambda}_{\mu} T^{\mu}_{\lambda}(N) \arrow[r] \arrow[d, "\wr\,\varphi_N"] \arrow[dr, phantom, "\circlearrowright"] & T^{\lambda}_{\mu} T^{\mu}_{\lambda}(L) \arrow[r] \arrow[d, "\wr\,\varphi_L"] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & L \arrow[r] & 0
\end{tikzcd}
$$

Use induction on length

φ . □

Translation to upper closures: simples again

Theorem

Let $\lambda, \mu \in \mathfrak{h}^*$ be antidominant and compatible. Assume that $\lambda^\sharp$ lies in a facet F of $E(\lambda)$, while $\mu^\sharp$ lies in $\overline{F}$.

1. For all $w \in W_{[\lambda]}$, if $w \cdot \mu^\sharp$ lies in the upper closure of $w \cdot F$, then $T_\lambda^\mu L(w \cdot \lambda) \cong L(w \cdot \mu)$.
2. If $w \in W_{[\lambda]}$ but $w \cdot \mu^\sharp \notin w \cdot \widehat{F}$, then $T_\lambda^\mu L(w \cdot \lambda) = 0$.

- $M(w \cdot \lambda)$ has composition series.

$T_\lambda^\mu (M(w \cdot \lambda))$ has composition series

$\shortparallel$

$M(w \cdot \mu)$

quotients $T_\lambda^\mu (M(w'w \cdot \lambda))$

$w'w \cdot \lambda \leq w \cdot \lambda$

has unique occurrences of $L(w \cdot \mu)$

$\exists\ w' w\cdot\lambda \leq w\cdot\lambda$

$T_\lambda^\mu (L(w'w\cdot\lambda)) = L(w\cdot\mu)$

$\cong$

$L(w'w\cdot\mu)$

$\Rightarrow\ w'w\cdot\mu = w\cdot\mu$

$w' \in W^0_{w\cdot\mu}$ = group generated by reflections which fixes the facet $w\cdot F'$ which contains $w\cdot\mu$

$\alpha \in \Phi^0_{w\cdot F'} = w\,\Phi^0_{F'}$.

- $w \cdot \mu^{\#} \in \widehat{w \cdot F}$

$$\langle w \cdot \lambda^{\#} + \rho, \alpha^{\vee} \rangle \leq 0 \quad \forall \alpha \in \Phi^{0}_{w.F'} \subseteq \Phi^{0}_{w.F} \cup \Phi^{-}_{w.F}$$

$$\Leftrightarrow \quad s_{\alpha}(w \cdot \lambda^{\#}) \geq w \cdot \lambda^{\#}$$

0.6

$$\Leftrightarrow \quad w'(w \cdot \lambda^{\#}) \geq w \cdot \lambda^{\#} \qquad w' \in W^{0}_{w \cdot \mu}$$

$$\Rightarrow \quad w' w \cdot \lambda \geq w \cdot \lambda \geq w' w \cdot \lambda.$$

$$\Rightarrow \quad w' w \cdot \lambda = w \cdot \lambda$$

$$T^{\mu}_{\lambda}(L(w \cdot \lambda)) = L(w \cdot \mu).$$

. $w \cdot \mu^{\#} \notin \widehat{w \cdot F}$

$\exists\, \alpha \in \Phi_F^+$

$\langle w \cdot \lambda^{\#} + \rho, \alpha^\vee \rangle > 0$

and $\langle w \cdot \mu^{\#} + \rho, \alpha^\vee \rangle = 0.$

$\Longleftrightarrow$ $\boxed{s_\alpha (w \cdot \lambda^{\#}) < w \cdot \lambda^{\#}}$ and $s_\alpha \cdot (w \cdot \mu^{\#}) = w \cdot \mu^{\#}.$

$0 \to M(s_\alpha (w \cdot \lambda)) \to M(w \cdot \lambda) \to Q \to 0$

apply T_λ^μ $\qquad$ $L(w \cdot \lambda)$ is a quotient of Q.

$0 \to M(s_\alpha(w \cdot \mu)) \to M(w \cdot \mu) \to T_\lambda^\mu (Q) \to 0$

$\quad \shortparallel \; M(w \cdot \mu)$

Conclude

$$T_\lambda^\mu(Q) = 0 \quad \text{then also} \quad T_\lambda^\mu(L(w\cdot\lambda)) = 0$$

$\square$.

Exercise

Let λ, μ satisfy the hypotheses of the above theorem. Prove that for all $w \in W_{[\lambda]}$, the projective module $T_\lambda^\mu P(w \cdot \lambda)$ is nonzero.

$$P(w\cdot\lambda) \twoheadrightarrow M(w\cdot\lambda)$$

apply T_λ^μ $\qquad \downarrow$

$$T_\lambda^\mu(P(w\cdot\lambda)) \twoheadrightarrow T_\lambda^\mu(w\cdot\lambda) = M(w\cdot\mu)$$

$$\neq 0$$

Character formula

$b^{\lambda}_{ww} = 1$

$$[L(w\cdot\lambda)] = \sum_{w'\in W_{[\lambda]},\, w'\cdot\lambda\uparrow w\cdot\lambda} b^{\lambda}_{w',w}[M(w'\cdot\lambda)]$$

T_{λ}^{μ}

$\mu^{\#}, \lambda^{\#}$ are in the same facet

$$[L(w\cdot\mu)] = \sum_{w'\in W_{[\mu]},\, w'\cdot\lambda\uparrow w\cdot\lambda} b^{\lambda}_{w',w}[M(w'\cdot\mu)]$$

$\mu^{\#}$ is the upper closure of facet containing $\lambda^{\#}$.

$[L(w\cdot\mu)]$ $\quad$ $[M(w'\cdot\mu)]$

$$\sum_{w'z \leq w} e^{\lambda}_{w'z, w}$$

λ is regular

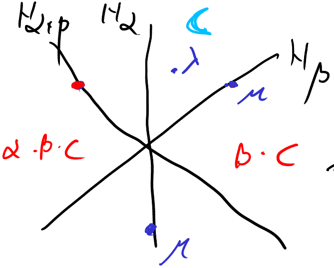
$z \in W^{\circ}_{\mu}$

$w' z \cdot \mu = w' \cdot \mu$

Example: $\mathfrak{sl}_3(\mathbb{C})$

(Section 5.4) $\lambda \in \Lambda$ regular and antidominant, $\mu \in \Lambda$, antidominant and lies in β-hyperplane. $w = s_\alpha s_\beta$. Then

$$[L(w \cdot \lambda)] = [M(w \cdot \lambda)] - [M(s_\alpha \cdot \lambda)] - [M(s_\beta \cdot \lambda)] + [M(\lambda)].$$



apply T_λ^μ

$s_\beta \mu = \mu$

$w \cdot \mu \in \widehat{w \cdot C}$

$$[L(w\cdot\mu)] = [M(w\cdot\mu)] - [M(s_\alpha\cdot\mu)] - [\cancel{M(s_\beta\cdot\mu)}] + [\cancel{M(\mu)}].$$

$$\underset{M(\mu)}{\parallel}$$

Projective modules

Theorem

Let $\lambda, \mu \in h^*$ be antidominant and compatible. Further assume that $\mu^\sharp$ lies in the closure of the facet F to which $\lambda^\sharp$ belongs. If $w \cdot \mu^\sharp$ lies in $\widehat{w \cdot F}$ for some $w \in W_{[\lambda]}$, then $T_\mu^\lambda(P(w \cdot \mu)) \cong P(w \cdot \lambda)$.

We will use Theorem 3.9(b): When a projective module $P \in \mathcal{O}$ is written as a direct sum of indecomposables, the number of those isomorphic to $P(\lambda)$ is equal to $\dim \mathrm{Hom}_{\mathcal{O}}(P, L(\lambda))$.