Translation functors

Sigiswald Barbier

Ghent University

Category $\mathcal{O}$ is decomposable:

$$\mathcal{O} = \bigoplus_{\chi} \mathcal{O}_\chi$$

Theorem (Theorem 4.6)

The blocks of $\mathcal{O}$ are precisely the subcategories consisting of modules whose composition factors all have highest weights linked by $W_{[\lambda]}$ to an antidominant weight λ. Thus the blocks are in natural bijection with antidominant (or alternatively, dominant) weights.

Denote the block associated to an antidominant weight λ by $\mathcal{O}_\lambda$.

$T : \mathcal{O}_0 \to \mathcal{O}_\lambda$

Translation functor T_λ^μ

T : simple → simple
Verma → Verma
projectives → projectives.
equivalence between category.

- projecting on $\mathcal{O}_\chi$ is exact
- tensoring with a finite dim module is exact.

$$T(M) = pr_\mu\left(\left(L \otimes pr_\lambda(M)\right)\right) \qquad \mathcal{O} \to \mathcal{O}$$

$$\mathcal{O}_\lambda \to \mathcal{O}_\mu$$

$$\mathcal{O}_{\chi_\lambda} = \mathcal{O}_\lambda$$

Translation functor T_λ^μ

$L = L(\bar{\nu})$

$\bar{\nu}$ as the dominant weight in the W-orbit of $\nu = \mu - \lambda \in \Lambda$

$$T_\lambda^\mu = pr_\mu\left(L(\bar{\nu}) \otimes pr_\lambda(\cdot) \right)$$

Overview

- Basic properties of T_λ^μ (7.1-7.2)
- Weyl group geometry
  - Facets (7.3)
  - Non-integral weights (7.4)
  - Key-lemma (7.5)
- Easy: translation from facet to facet closure
  - Verma modules (7.6)
  - Simple modules (7.7)
  - Categorical equivalences (7.8)
  - Simple modules again (7.9)
  - Characters (7.10)
- Hard: translations from facet closure to facet
  - Projective modules (7.11)
  - Translation from a wall (7.12-7.14)
  - Translation across a wall (7.15)
  - Self-dual projectives (7.16)

Basic properties

$L(\bar{V})$

Theorem

If $\lambda, \mu \in h^*$, with $\mu - \lambda \in \Lambda$, the exact functor T_λ^μ

- *commutes with the duality functor*
- *takes projective modules to projective modules.*
- *$T_\lambda^\mu(M(\lambda))$ has a standard filtration with $M(\mu)$ as a subquotient.*

$$\left\{\begin{array}{l} \bullet \quad pr_\lambda(M^\vee) = (pr_\lambda(M))^\vee \qquad (3.2\,(e)(d)) \\ \bullet \quad (L \otimes M)^\vee \quad L^\vee \otimes M^\vee \text{ if } \dim L < \infty \quad (\text{exercise } 3.2) \\ \bullet \quad L^\vee = L \quad \text{if } L \text{ is simple.} \end{array}\right.$$

$$(T_\lambda^\mu(M))^\vee = T_\lambda^\mu(M^\vee)$$

$\to$ $L \otimes P$ is projective for P projective
L finite dim.

$\exists$ $L(\bar{\nu}) \otimes M(\lambda)$ has a standard filtration with $\lambda + \nu'$ with ν' a weight of $L(\bar{\nu})$

$\dim (L(\bar{\nu}))_{\nu'} = (L(\bar{\nu}) \otimes M(\lambda) : M(\lambda + \nu'))$

$\nu' = w\bar{\nu}$ $\Rightarrow \dim L(\bar{\nu})_{\nu'} = 1$ (1.6)

$\Rightarrow M(\lambda + \nu)$ will be a subquotient

$\lambda + \nu = \lambda + \mu - \lambda = \mu$

Theorem (Adjoint functor property)

Let $\lambda, \mu \in h^*$ be compatible. Then T_λ^μ is left and right adjoint to T_μ^λ:

$$\mathrm{Hom}_{\mathcal{O}}(T_\lambda^\mu(M), N) \cong \mathrm{Hom}_{\mathcal{O}}(M, T_\mu^\lambda(N))$$

for all $M, N \in \mathcal{O}$.

Use a lemma: $\mathrm{Hom}_{\mathcal{O}}(L \otimes M, N) \cong \mathrm{Hom}_{\mathcal{O}}(M, L^* \otimes N)$

Proof $M \in \mathcal{O}_\lambda$, $\in \mathcal{O}_\mu$

$$\mathrm{Hom}_{\mathcal{O}}(L(\bar{\nu}) \otimes M, N) \cong \mathrm{Hom}_{\mathcal{O}}(M, L(\overline{-\nu}) \otimes N)$$

$$\cong$$

$$\mathrm{Hom}_{\mathcal{O}}(M, L(\bar{\nu})^* \otimes N)$$

Need to show is that $L(\bar{\nu})^* \cong L(\overline{\lambda-\mu})$

$L(\bar{\nu})^* = L(-w_0\bar{\nu})$ 1.6.

$\overline{-\nu}$ is the dominant weight in W-orbit of $-\nu = \lambda - \mu$

$$\overline{v} = wv$$

$$-w_0 \overline{v} = -w_0 w v = w_0 w(-v)$$

$$\overline{-v} = w_0 w(-v)$$

Exercise

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Show that T_λ^μ need not take Verma modules to Verma modules. For example, let $\lambda = 1$ and $\mu = -3$.

$\nu = -4$, $\bar{\nu} = 4$

$L(\bar{\nu})$ is 5-dim $v_4 - v_2 - v_0 - v_{-2} - v_{-4}$

$L(\bar{\nu}) \otimes M(1)$ has a standard filtration with factor $M(5), M(3), \underline{M(1)}, M(-1), \underline{M(-3)}$

Project on $\mathcal{O}_\mu = \mathcal{O}_{-3}$

$-3, 1 \Rightarrow M(-3), M(1) \in \mathcal{O}_{-3}$

$0 \to M(1) \to T_\lambda^\mu(M(1)) \twoheadrightarrow M(-3) \to 0$

Facets

Definition

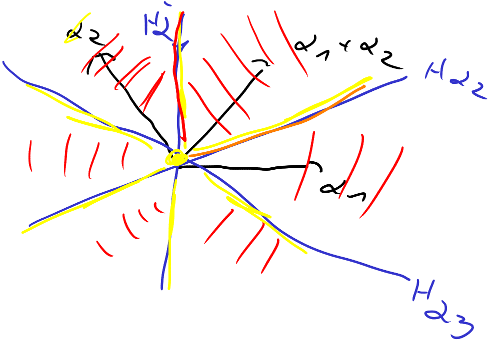
a facet F is a nonempty subset of E determined by a partition of ϕ^+ into disjoint subsets ϕ_F^0, ϕ_F^+, ϕ_F^- :

$$\lambda \in F \Leftrightarrow \begin{cases} \langle \lambda + \rho, \alpha^\vee \rangle = 0 & \text{when } \alpha \in \phi_F^0, \\ \langle \lambda + \rho, \alpha^\vee \rangle > 0 & \text{when } \alpha \in \phi_F^+, \\ \langle \lambda + \rho, \alpha^\vee \rangle < 0 & \text{when } \alpha \in \phi_F^-, \end{cases}$$

Clearly the closure $\overline{F}$ is obtained by replacing $>$ by $\geq$ and $<$ by $\leq$.

$\rho = -1$

$\alpha \to H_\alpha$



Chamber

if $\Phi_F^0 = \emptyset$

wall

$|\Phi_F^0| = 1$

Upper closure

Definition

The upper closure $\widehat{F}$ of the facet F is defined by the conditions

$$\lambda \in \widehat{F} \Leftrightarrow \begin{cases} \langle \lambda + \rho, \alpha^\vee \rangle = 0 & \text{when } \alpha \in \phi_F^0, \\ \langle \lambda + \rho, \alpha^\vee \rangle > 0 & \text{when } \alpha \in \phi_F^+, \\ \langle \lambda + \rho, \alpha^\vee \rangle \leq 0 & \text{when } \alpha \in \phi_F^-, \end{cases}$$

Exercise

Let F be a facet in E.

- *Prove that $\overline{F}$ is the union of the facets F' for which $\phi^+_{F'} \subset \phi^+_F$ and $\phi^-_{F'} \subset \phi^-_F$.*
- *Prove that $\widehat{F}$ is the union of the facets $F' \subset \overline{F}$ satisfying for every $\alpha > 0$ the condition:*

$$\langle \lambda + \rho, \alpha^\vee \rangle \leq 0 \text{ for all } \lambda \in F' \Leftrightarrow \langle \lambda + \rho, \alpha^\vee \rangle \leq 0 \text{ for all } \lambda \in F.$$

Theorem

Upper closures of facets have the following properties:

- *Each facet lies in the upper closure of a unique chamber.*
- *If the facet F lies in the upper closure of the chamber C, then $\overline{F} \subset \overline{C}$.*

- Define C by
$$\Phi_C^+ = \Phi_F^+ , \quad \Phi_C^- = \Phi_F^- \cup \Phi_F^0$$

- Trivial.

Non-integral weights

$$\Phi_{[\lambda]} = \{ \alpha \in \Phi \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z} \}$$

$$W_{[\lambda]} = \{ w \in W \mid w \cdot \lambda - \lambda \in \Lambda_r \}$$

$$= \langle s_\alpha \mid \alpha \in \Phi_{[\lambda]} \rangle$$

$E(\lambda)$ generated by roots of $\Phi_{[\lambda]}$.

Restrict λ to $E(\lambda)$

$\lambda^\#$ in $E(\lambda)$

$\langle \lambda^\# + \rho, \alpha^\vee \rangle = \langle \lambda + \rho, \alpha^\vee \rangle$ for all $\alpha \in \Phi_{[\lambda]}$

Proposition

Let $\lambda \in h^*$, with stabilizer W_λ° in $W_{[\lambda]}$

- For all $w \in W_{[\lambda]}$, we have $\lambda - w \cdot \lambda = \lambda^\sharp - w \cdot \lambda^\sharp$, while $(w \cdot \lambda)^\sharp = w \cdot \lambda^\sharp$.
- The stabilizer of $\lambda^\sharp$ in $W_{[\lambda]}$ is W_λ°.
- Suppose $\phi_{[\lambda]} = \phi_{[\mu]} = \phi_{[\lambda+\mu]}$. Then $(\lambda + \mu)^\sharp = \lambda^\sharp + \mu^\sharp$.

$$\cdot \quad \lambda - s_\alpha \cdot \lambda = \langle \lambda + \rho, \alpha^\vee \rangle \alpha$$
$$= \langle \lambda^\# + \rho, \alpha^\vee \rangle \alpha$$
$$= \lambda^\# - w \cdot \lambda^\#$$

$$\langle s_\alpha \cdot \lambda + \rho, \beta^\vee \rangle = \langle s_\alpha \cdot \lambda^\# + \rho, \beta^\vee \rangle$$

$$\Leftrightarrow \langle (\lambda + \rho), s_\alpha \beta^\vee \rangle = \langle (\lambda^\# + \rho), s_\alpha \beta^\vee \rangle. \quad \square$$

- s_α is stabilizer of $\lambda^\#$

$$\langle \lambda^\# + \rho, \alpha^\vee \rangle = 0$$

$$\Rightarrow \langle \lambda + \rho, \alpha^\vee \rangle = 0$$

$\Rightarrow$ s_α is a stabilizer of λ.

- $\langle \lambda + \mu + \rho, \alpha^\vee \rangle$

$$= \langle \lambda + \rho, \alpha^\vee \rangle + \langle \mu, \alpha^\vee \rangle$$

$$= \langle \lambda^\# + \cancel{\rho}, \alpha^\vee \rangle + \langle \mu^\# + \rho, \alpha^\vee \rangle$$

$$= \langle \lambda^\# + \mu^\# + \rho, \alpha^\vee \rangle$$

Exercise

Show that W_λ° is the group which fixes pointwise the facet $F \subset E(\lambda)$ to which $\lambda^\sharp$ belongs; in turn, W_λ° is the group fixing $\overline{F}$ pointwise.

Key lemma

$\mu - \lambda \in \Lambda$

Lemma

Let $\lambda, \mu \in$ h* be compatible, with $\nu := \mu - \lambda \in \Lambda$ and $\overline{\nu}$ the unique W-conjugate of ν lying in Λ^+. Assume that $\lambda^\sharp$ lies in a facet F of E(λ), while $\mu^\sharp$ lies in the closure $\overline{F}$. Then for all weights $\nu' \neq \nu$ of L$(\overline{\nu})$, the weight $\lambda + \nu'$ is not linked by W$_{[\lambda]}$ = W$_{[\mu]}$ to $\lambda + \nu = \mu$.

Proof: $w \in W_{[\mu]}$ s.t.

$$w \cdot (\lambda + \nu') = \lambda + \nu$$

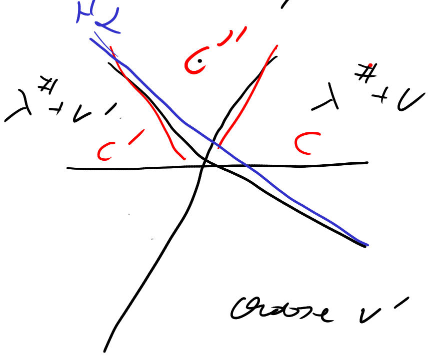
$$\Leftrightarrow \quad w \cdot \lambda + w\nu' = \lambda + \nu$$

$$\Leftrightarrow \quad \lambda - w \cdot \lambda = w\nu' + \nu$$

$$\Leftrightarrow \quad \lambda^\# - w \cdot \lambda^\# = w\nu' + \nu$$

$$\Leftrightarrow \quad w \cdot (\lambda^\# + \nu') = \lambda^\# + \nu$$

Assume $v' \neq v$ s.t. $w \cdot (\lambda^\# + v') = \lambda^\# + v$



$d(C, C')$

as number of root hyperplanes between these chambers.

Choose v' s.t. $d(C, C')$ is minimal.

$\lambda^\# + v \in \bar{C}$

$\lambda^\# + v' \in \bar{C}'$

If $d(C, C') = 0$ not possible

since C is fundamental domain of $W_{[\lambda]}$

2] $d(C, C') > 0$, then choose α

s.t. H_α separates C and C'

and $H_\alpha \cap \overline{C}' \neq \phi$.

C' is on positive side of H_α

C neg " H_α.

$\Rightarrow \langle \lambda^{\#} + \nu' + \rho, \alpha^\vee \rangle \geq 0$

$\langle \underbrace{\lambda^{\#} + \nu}_{\mu^{\#}} + \rho, \alpha^\vee \rangle \leq 0.$

3] $C'' = s_\alpha C'$ $\quad d(C'', C) < d(C', C)$

4) $\langle \lambda^{\#} + \nu' + \rho, \alpha^\vee \rangle \geq 0$

5) $\langle \xi + \rho, \alpha^\vee \rangle \leq 0$ for all $\xi \in \overline{C}$

$\langle \lambda^{\#} + \rho, \alpha^\vee \rangle \leq 0$ $\lambda^{\#} \in \overline{C}$

6) $s_\alpha \cdot (\lambda^{\#} + \nu') = \lambda^{\#} - \langle \lambda^{\#} + \rho, \alpha^\vee \rangle \alpha + \boxed{s_\alpha \nu'}$

$\leq \lambda^{\#} + \nu'$

7) $s_\alpha \nu' \leq \underbrace{s_\alpha \nu' - \langle \lambda^{\#} + \rho, \alpha^\vee \rangle \alpha}_{\nu''}$

$\leq \nu'$

8) $\boxed{s_\alpha \nu' \leq \nu'' \leq \nu'}$

$$s_\alpha \cdot (\lambda^\# + \nu') = \lambda^\# + \nu''$$

$$\parallel$$

$$\lambda^\# + \nu' - \langle \lambda^\# + \nu' + \rho, \alpha^\vee \rangle \alpha$$

9] $s_\alpha \nu', \nu'$ are weights of $L(\bar{V})$

so ν'' is also a weight of $L(\bar{V})$ (1.6)

1.9] $s_\alpha(\lambda^\# + \nu') = \lambda^\# + \nu'' \in C''$

$d(C, C'') < d(C, C')$

and $\lambda^\# + \nu''$ is W-connected to $\lambda^\# + \nu$

$\Rightarrow \nu = \nu''$

11) $s_\alpha \nu' \le \nu \le \nu'$

$ = \ne$

$\nu+\alpha,\ \nu-\alpha$ are ~~not~~ not dominant weights of $L(\bar{V})$ (1.6)

$s_\alpha \nu' = \nu = \nu''$

12) $s_\alpha \nu' = s_\alpha \nu' - \underbrace{\langle \lambda^\# + \rho, \alpha^\vee \rangle \alpha}_{=0}$

$\Rightarrow \langle \lambda^\# + \rho, \alpha^\vee \rangle = 0$

13) $\alpha \in \bar{\Phi}_F^0 \to \langle \xi + \rho, \alpha^\vee \rangle = 0 \quad \forall \xi \in \bar{F}$

$\langle \lambda^\# + \nu + \rho, \alpha^\vee \rangle = 0$

$\Rightarrow \langle \nu, \alpha^\vee \rangle = 0$.

경우 2 $v' = v$ and $\langle v, \alpha^\vee \rangle = 0$ ↯

$v' \neq v$

Exercise

Exercise. How does the proof simplify if both $\lambda^\sharp$ and $\mu^\sharp$ are assumed to lie in C?