Humphrey's : BGG resolutions

Chapter 6: Extensions and Resolutions

§ 6.1

Fix $\lambda \in \Lambda^+$. Want to realise

$$\operatorname{ch} L(\lambda) = \sum_{w} (-1)^{\ell(w)} \operatorname{ch} M(w.\lambda) \qquad (2.4)$$

Def A BGG resolution of $L(\lambda)$ is an exact sequence

$$(*) \quad 0 \to C_m \xrightarrow{\delta_m} C_{m-1} \to \cdots \to C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\varepsilon} L(\lambda) \to 0$$

with $\displaystyle C_k := \bigoplus_{w \in W^{(k)}} M(w.\lambda)$, $k = 0, 1, \dots, m = |\Phi^+|$

$(C_m = M(w_0.\lambda)$, $C_0 = M(\lambda))$.

Goals:
(1) BGG resolutions exist
(2) Uniqueness ?
(3) Applications (LA cohomology, Homology of $\mathcal{O}$).

Exercise: $(C_\bullet, \delta_\bullet) \to L(\lambda)$ BGG resol $\Rightarrow \delta|M(w.\lambda) \neq 0$

Sketch: $\lambda \in \Lambda^+ \Rightarrow w_0 \cdot \lambda = \mu$ is anti dom regular.

$M(w.\lambda) = \langle v_+ \rangle$, $w \in W^{(k)}$

$\delta(v_+) = 0 \Rightarrow \exists\, v \in C_{k+1}$ s.t. $\delta(v) = v_+$

$\Rightarrow C_{k+1} \twoheadrightarrow M(w.\lambda)$

$\Rightarrow [M(u.\lambda) : L(w.\lambda)] \neq 0$, $\exists\, u \in W^{(k+1)}$

$\Rightarrow M(w.\lambda) \hookrightarrow M(u.\lambda)$

$\Leftrightarrow ww_0 \cdot \mu \leq uw_0 \cdot \mu$

$\Leftrightarrow ww_0 \leq uw_0$

$\Rightarrow k = \ell(w) \geq \ell(u) = k+1$ □

§ 6.2

Thm $\lambda \in \Lambda^+$. There is an exact seq. $(D^\lambda_\bullet, \partial_\bullet) \to L(\lambda)$ s.t. D^λ_k has std filt with $(D^\lambda_k : M(w\cdot\lambda)) = 1$, $\forall w \in W^{(k)}$.

Sketch: Assume $\lambda = 0$

$\mathfrak{b}$-cosets: $v_j = y_j + \mathfrak{b}$

(A) Let $\mathfrak{n}^- \cong V = \mathfrak{g}/\mathfrak{b} \supseteq \{v_1, \dots, v_m\}$ basis

Wts V : $-\alpha_i \leftrightarrow v_i$

Wts $\Lambda^k V$: $-\sum \alpha_{i_j} \leftrightarrow v_{i_1} \wedge \dots \wedge v_{i_k}$

(B) $D_k := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\Lambda^k V)$, has std filt. (3.6)

$\Rightarrow D_0 = M(0)$ $\quad (\Lambda^0 V = \text{triv})$

$D_m = M(w_0 \cdot 0)$ $\quad (h.v_1 \wedge \dots \wedge v_m = -2\rho(h)\, v_1 \wedge \dots \wedge v_m)$

(C) Introduce $\partial_k : D_k \to D_{k-1}$, ε (general construction)

(D) $D_k^0 := D_k \cap \mathcal{O}^{\chi_0}$ (principal block)

(E) Apply T_0^λ to pass from $L(0) \to L(\lambda)$ $\square$

Details are in § 6.3 – § 6.5

Claim Each D_k has std filtration

Pf: $M = \Lambda^k V \supseteq \{z_1, \dots, z_N\}$

$\mathrm{Wts}(M) \supseteq \{\mu_1, \dots, \mu_N\}$

$\mu_i \leqslant \mu_j \Rightarrow i \leqslant j$

$$\rightsquigarrow \quad 0 \subseteq M_N \subseteq \cdots \subseteq M_2 \subseteq M_1 = D_k$$

$$M_j := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \langle z_j, \dots, z_N \rangle$$

s.t. $M_j / M_{j+1} \cong M(\mu_j)$ $\square$

Example $\mathfrak{g} = \mathfrak{sl}(3)$, $\Phi^+ = \{ \alpha, \beta, \gamma = \alpha+\beta \}$

$$V \supseteq \{ v_\alpha, \quad v_\beta, \quad v_\gamma \}$$

wts: $-\alpha \qquad -\beta \qquad -\alpha-\beta$

$$M := \Lambda^2 V \supseteq \{ v_\alpha \wedge v_\gamma, \quad v_\beta \wedge v_\gamma, \quad v_\alpha \wedge v_\beta \}$$

wts:
$$\begin{tikzcd} -(2\alpha+\beta) \arrow[rr, bend right, "+\alpha"'] & -(\alpha+2\beta) & -(\alpha+\beta) \end{tikzcd}$$

Now: $\mu_3 - \mu_1 = \alpha > 0$

$$X_\alpha (v_\alpha \wedge v_\gamma) = X_{\underline{\alpha}} v_\alpha \wedge v_\gamma + v_\alpha \wedge X_{\underline{\alpha}} v_\gamma \in \langle v_\alpha \wedge v_\beta \rangle$$

$$X_\beta (v_\alpha \wedge v_\gamma) = X_\beta v_\alpha \wedge v_\gamma + v_\alpha \wedge X_\beta v_\gamma = 0$$

$$X_\gamma (v_\alpha \wedge v_\gamma) = X_\gamma v_\alpha \wedge v_\gamma + v_\alpha \wedge X_\gamma v_\gamma = 0$$

$\therefore$ $v_\alpha \wedge v_\gamma$ is HWV modulo M_2.

Further $W^{(2)} \cdot 0 = \{ s_\alpha s_\beta \cdot 0 = -(2\alpha+\beta),$
$\qquad s_\beta s_\alpha \cdot 0 = -(\alpha+2\beta) \}$

$\therefore$ $W^{(k)} \cdot 0 \neq \mathrm{Wts}(\Lambda^k V)$

Recall $M = \bigoplus_{\chi} M^{\chi} \quad \forall M \in \mathcal{O}$

Claim: $M, N \in \mathcal{O}$, $\varphi \in \mathrm{Hom}_{\mathcal{O}}(M,N) \Rightarrow \varphi(M^{\chi}) \subseteq N^{\chi}$

Pf: $v \in M^{\chi}$, $z \in Z(\mathfrak{g}) \Rightarrow (z - \chi(z))^n v = 0, \ \exists n$

$\Rightarrow 0 = (z-\chi(z))^n \varphi(v)$ □

$\therefore \ (D_\bullet, \partial_\bullet) \to L(\lambda)$ exact $\Rightarrow (D_\bullet^{\chi_\lambda}, \partial_\bullet^{\chi_\lambda}) \longrightarrow L(\lambda)$ exact

Def: $\Pi_w := \Phi^+ \cap w(\Phi^-)$

$\Gamma_w := \Phi^+ \cap w(\Phi^+)$

Notation: $\Pi \subseteq \Phi^+ \Rightarrow \overline{\Pi} = \sum_{\alpha \in \Pi} \alpha \in \mathfrak{h}^*$.

Lemma: $\mu = w \cdot 0$ occurs in $\Lambda^{\ell(w)} V$

Pf: $\Phi^+ = \Phi^+ \cap (w\Phi^+ \cup w\Phi^-)$

$= \Gamma_w \cup \Pi_w$

$w\Phi^+ = w\Phi^{\overset{-}{+}} \cap (\Phi^+ \cup \Phi^{\overset{+}{-}})$

$= \Gamma_w \cup (-\Pi_w)$

$\Rightarrow w \cdot 0 = w\rho - \rho$

$= \frac{1}{2}(\overline{\Gamma_w} - \overline{\Pi_w}) - (\overline{\Gamma_w} + \overline{\Pi_w})$ □

$= -\overline{\Pi_w}$

Lemma $\mu = w.0$ occurs only once in $\Lambda^{\bullet} V$

Pf: We show: $\Pi \subset \Phi^+$ s.t. $\overline{\Pi} = \overline{\Pi}_w \Rightarrow \Pi = \Pi_w$.

Clear $\ell(w) = 0$

Suppose $\ell(w) = k > 0$

$\Rightarrow \ell(s_\alpha w) = k-1$, $\exists \alpha \in \Delta$

(0.3) $\Rightarrow w^{-1}\alpha < 0$

$\Rightarrow \begin{cases} \alpha \in \Pi_w \\ (w')^{-1}\alpha > 0 \end{cases}$ $\qquad w' = s_\alpha w$

$\Rightarrow \underline{\alpha \notin \Pi_{w'}}$

Claim $\Pi_w = s_\alpha \Pi_{w'} \cup \{\alpha\}$ (*)

Pf: Have

$$s_\alpha \Pi_{w'} = s_\alpha(\Phi^+ \cap w'\Phi^-)$$

$$= (\Phi^+ \setminus \{\alpha\} \cup \{-\alpha\}) \cap \overset{s_\alpha w'}{w}\Phi^-$$

$(\alpha \in \Pi_w \subseteq w\Phi^-)$ $= (\Phi^+ \setminus \{\alpha\}) \cap w\Phi^-$

$\Rightarrow \{\alpha\} \cup s_\alpha(\Pi_{w'}) = \Phi^+ \cap w\Phi^- = \Pi_w$. □

Back to the Lemma:

$$\Pi \subseteq \Phi^+, \quad \overline{\Pi} = \Pi_w = \rho - w\rho$$

$$\Rightarrow \underline{s_\alpha \overline{\Pi}} = (\rho - \alpha) - s_\alpha w \rho = (\rho - w'\rho) - \underline{\alpha}$$

$$= \underline{\overline{\Pi}_{w'} - \alpha}$$

$$\alpha \notin \Pi \Rightarrow s_\alpha \Pi \subseteq \Phi^+$$

$$\Rightarrow s_\alpha \Pi \cup \{\alpha\} \subseteq \Phi^+$$

$$\Rightarrow \overline{s_\alpha \Pi \cup \{\alpha\}} = \overline{\Pi_{w'}}$$

$$\text{(IH)} \Rightarrow \Pi_{w'} = s_\alpha \Pi \cup \{\alpha\}$$

$$\Rightarrow \alpha \in \Pi_{w'} \qquad \text{(contr.)}$$

$$\therefore \alpha \in \Pi$$

Let $\Pi' = s_\alpha(\Pi \setminus \{\alpha\}) \subseteq \Phi^+$

$$\Rightarrow \overline{\Pi'} = \overline{\Pi_{w'}} \qquad \overline{\Pi} = \Pi_w$$

$$\text{(IH)} \Rightarrow \underline{\Pi'} = \underline{\Pi_{w'}}$$

$$\Rightarrow \Pi = s_\alpha(\Pi_{w'}) \cup \{\alpha\} = \Pi_w \qquad \square$$

So far we showed:

- $\mathrm{Wts}(D_k^0) = \{ w \cdot 0, \; w \in W^{(k)} \}$
- D_k^0 has a std filtration.

<u>Question:</u> $\mathrm{Ext}_{\mathcal{O}}(M(w \cdot \lambda), M(w' \cdot \lambda)) = ?$

if $\lambda \in \Lambda^+$, $\ell(w) = \ell(w')$.

Thm (~~6.3~~) 6.5 $\lambda \in \mathfrak{h}^*$

(a) $\mathrm{Ext}_{\mathcal{O}}(M(\mu), M(\lambda)) \neq 0 \Rightarrow \mu \uparrow \lambda, \ \mu \neq \lambda$

(b) $\lambda \in \Lambda^+, \ w, w' \in W.$ Then

$$\mathrm{Ext}_{\mathcal{O}}(M(w\cdot\lambda), M(w'\cdot\lambda)) \neq 0 \Rightarrow w' < w$$

$$\Rightarrow \ell(w) < \ell(w')$$

Pf: (a) $(3.1\,a) \Rightarrow \mu \neq \lambda.$ Given

$$\begin{tikzcd}
 & & & P(\mu) \arrow[dl, "\varphi"'] \arrow[d, "\pi"] & & \\
0 \arrow[r] & M(\lambda) \arrow[r, "f"] & M \arrow[r, "g"] & M(\mu) \arrow[r] \arrow[l, bend left=60, dashed, "\sigma"] & 0 & (*)
\end{tikzcd}$$

if $v \in M(\mu)$, $x, y \in P(\mu)$ s.t. $v = \pi x = \pi y$

$$0 = \pi(x-y) = g\varphi(x-y)$$

$$\Rightarrow \varphi(x-y) \in \mathrm{im}(f) \Leftrightarrow \varphi(x-y) \in M(\lambda)$$

<u>if:</u> $\varphi P(\mu) \cap \mathrm{im} f = 0 \Rightarrow \sigma(v) = \varphi(x), \ \exists x \in \pi^{-1} v$

$\Rightarrow (*)$ splits

(3.10) $0 \subseteq P_{\underline{0}} \subseteq P_{\underline{1}} \subset \dots \subset P_{\underline{n}} = P(\mu), \ P_i / P_{i-1} \cong M(\mu_i)$

(3.11) $(P(\mu) : M(\mu_i)) = [M(\mu_i) : L(\mu)] > 0$

(5.1) $\underline{\mu \uparrow \mu_i}$ $\forall i$

$f(M(\lambda)) \cap \varphi(P(\mu)) \neq 0 \Rightarrow \varphi P_i \cap f(M(\lambda))$, $\exists i$ min.

$\Rightarrow \varphi|P_i : P_i \to M(\lambda)$

$\rightsquigarrow P_i / P_{i-1} \cong M(\mu_i)$

$\Rightarrow [M(\lambda) : L(\mu_i)] \neq 0 \Rightarrow \underline{\exists \mu_i \uparrow \lambda}.$ $\Rightarrow \mu \uparrow \lambda$

(b) $\lambda \in \Lambda^+ \Rightarrow \mu = w_0 \cdot \lambda$ is antidom regular

$\therefore \ \mathrm{Ext}_{\mathcal{O}}(M(w\cdot\lambda), M(w'\cdot\lambda)) \neq 0$

(a) $\Longrightarrow \ ww_0\cdot\mu \uparrow w'w_0\cdot\mu$ & $w \neq w'$

(5.2) $\Longrightarrow \ ww_0 < w'w_0$

$\Longrightarrow \ w > w'$ $\square$

General construction of δ_k's [Hilton-Stamb.]

From LA cohomology: M $\mathfrak{g}$-module, $V(\mathfrak{g}) = \text{triv}$

$$H^n(\mathfrak{g}, M) = \mathrm{Ext}^n_{\mathcal{U}}(V(\mathfrak{g}), M)$$

$\underline{P}$ proj. resol. of triv.

$$= H^n(\mathrm{Hom}_{\mathcal{U}}(\underline{P}, M))$$

Notations: $\underline{e}_k := e_1 \wedge \cdots \wedge e_k$ $\quad \exists\, e_1, \ldots, e_k \in \mathfrak{g}$

$\hat{\underline{e}}^{i}_k := e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_k$

$(i<j)$ $\quad \hat{\underline{e}}^{i,j}_k := e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_k$

Def: $P_k := \mathcal{U} \otimes \Lambda^k \mathfrak{g}$

$$\partial_k(u \otimes \underline{e}_k) = \sum_i (-1)^{i+1} u e_i \otimes \hat{\underline{e}}^i_k + \sum_{i<j} u \otimes [e_i, e_j] \wedge \hat{\underline{e}}^{i,j}_k$$

Prop: $\partial_{k-1} \partial_k = 0 \quad \forall k.$

Introduce filtrations on $\bigoplus_k P_k = \mathcal{U} \otimes \wedge \mathfrak{g}$, $q \geq 0$:

$F^q := \text{span}\{ \underline{e}_m \otimes \underline{e}_n \mid m+n = q \}$

PBW basis w.r.t. $\{ e_1, \dots, e_d \mid d = \dim \mathfrak{g} \}$

$$F^q P_n = F^q \cap P_n$$

Def: $\underline{W}^q = (W^q_\bullet , \partial^q_\bullet)$ a complex with.

$$W^q_n := F^q P_n / F^{q-1} P_n$$

$$\partial^q_n (u \otimes \underline{e}_n) = \partial_n (u \otimes \underline{e}_n) \mod F^{q-1}$$

$$\equiv \sum_i (-1)^{i+1} u e_{\ell_i} \otimes \hat{\underline{e}}^i_n$$

Thm: $\underline{W}^q$ exact $\forall q \geq 0$

Cor: $\underline{P} = (P_\bullet , \partial_\bullet)$ is a free-resol. of triv.

Pf: From SES $F^{q-1} \underline{P} \hookrightarrow F^q \underline{P} \twoheadrightarrow \underline{W}^q$

$H_n(\underline{W}^P) = 0 \overset{LES}{\Rightarrow} H_n(F^{q-1}\underline{P}) \cong H_n(F^q \underline{P})$, $\forall n$

$F^0 \underline{P} = 0 \to V(\mathfrak{g}) \to V(\mathfrak{g}) \to 0$

$\Rightarrow H_n(F^0 \underline{P}) = 0 \quad \forall n$

(induction) $\Rightarrow H_n(F^q \underline{P}) = 0 \quad \forall n, q$

$\Rightarrow H_n(\underline{P}) = 0$ □

Moral of the story: all goes through relatively [BGG]

The relative version for $(\mathfrak{g}, \mathfrak{b})$:

$$D_k = \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{b})} \Lambda^k(\mathfrak{g}/\mathfrak{b})$$

$$\delta_k(u \otimes \underline{v}_k) = \sum_i (-1)^i u e_i \otimes \hat{\underline{v}}_k^i + \sum_{i,j} (-1)^{i+j} u \otimes \overbrace{[e_i, e_j]}^{\text{in } \mathfrak{g}/\mathfrak{b}} \wedge \hat{\underline{v}}_k^{i,j}$$

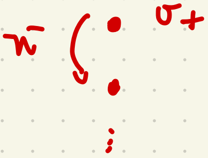
with $\underline{v}_k = v_1 \wedge \dots \wedge v_k \in \Lambda^k(\mathfrak{g}/\mathfrak{b})$

$e_i \in \mathfrak{g}$ a represent. of v_i

Is an exact complex. □

§6.6. Thm (Bott) $\lambda \in \Lambda^+$: $\dim H^k(\mathfrak{n}^-, L(\lambda)) = |W^{(k)}|$

Sketch: $H^k(\mathfrak{n}^-, L(\lambda)) = \mathrm{Ext}^k_{\mathfrak{n}_-}(\mathbb{C}, L(\lambda))$



$$\cong \mathrm{Ext}^k_{\mathfrak{n}_-}(L(\lambda)^\vee, \mathbb{C}^\vee)$$

take $\underline{M}$ to be BGG resol!

$$= H^k(\mathrm{Hom}_{\mathfrak{n}-}(\underline{M}, \mathbb{C}))$$

$\underline{M}$ any $\mathcal{U}(\mathfrak{n}^-)$-proj. resol. of $L(\lambda)$.

$\Rightarrow \mathrm{Hom}_{\mathfrak{n}-}(M(\mu), \mathbb{C}) \cong (M(\mu)/\mathfrak{n}^- M(\mu))^* \cong \mathbb{C}_{-\mu}$

$\Rightarrow \mathrm{Hom}_{\mathfrak{n}-}(D_k^\lambda, \mathbb{C}) \cong \bigoplus_{w \in W^{(k)}} \mathbb{C}_{-w.\lambda}$

$\Rightarrow H^k(\mathrm{Hom}_{\mathfrak{n}-}(\underline{C}_\bullet, \mathbb{C})) \cong \bigoplus_{w \in W^{(k)}} \mathbb{C}_{-w.\lambda}$ □

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}(D_k^{\lambda}, \mathbb{C}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Hom}(D_{k-1}^{\lambda}, \mathbb{C}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \cdots \\
\cdots \arrow[r, "0"] & \bigoplus_{w \in W^{(k)}} \mathbb{C}_{-w\cdot\lambda} \arrow[r, "0"] & \bigoplus_{u \in W^{(k-1)}} \mathbb{C}_{-u\cdot\lambda} \arrow[r, "0"] & \cdots
\end{tikzcd}$$

Remarks on Uniqueness of BGG-resolutions (6.7, 6.8).

Let $\underline{C} = (\underline{C}_\bullet, \delta_\bullet) \xrightarrow{\varepsilon} L(\lambda)$ be a BGG-res

Rewrite it as $\underline{C} = (C^\circ_\bullet, \varepsilon_\bullet)$

$$C^\circ_k = \bigoplus_{w \in W^{(k)}} M(w\cdot\lambda^\circ) \quad , \quad \lambda^\circ := w_0\cdot\lambda$$

$$\varepsilon_k : C^\circ_k \longrightarrow C^\circ_{k+1}$$

(Note: $C^\circ_k = C_{m-k}$, $\varepsilon_k = \delta_{m-k}$)

Now:

$$\varepsilon_k\big|_{M(w\cdot\lambda^\circ)} \neq 0 \Rightarrow M(w\cdot\lambda^\circ) \hookrightarrow M(w'\cdot\lambda^\circ),$$
$$\Leftrightarrow \quad w < w'$$

Notation / Def

$w \longrightarrow w'$: map $M(w\cdot\lambda^\circ) \hookrightarrow M(w'\cdot\lambda^\circ)$

$w \xrightarrow{\alpha} w'$: when $w' = s_\alpha w$, $\exists\, \alpha > 0$

Remarks:

(1) $w \to w'$ is defined up to a scalar call it $e(w,w') \in \mathbb{C}$

(2) $e(w,w') = 0$ if $\nexists\; w \longrightarrow w'$

(3) $\underline{C} = (C^{\circ}_{\bullet}, \varepsilon_{\bullet})$ defines a matrix

$$E = \left(e(w,w') \right)_{w,w' \in W}$$

Def: Let $(w_1, w_2, w_3, w_4) \in W^4$ s.t.

$$\begin{tikzcd}
& w_2 \arrow[dr] & \\
w_1 \arrow[ur] \arrow[dr] & & w_4 \\
& w_3 \arrow[ur] &
\end{tikzcd}$$

These elements are said to form a square.

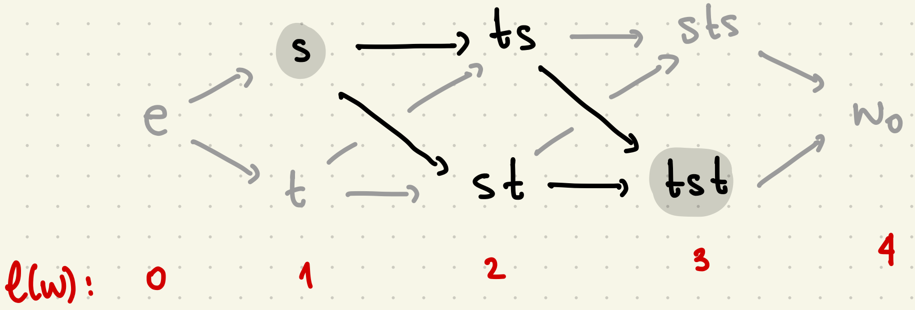
Fact: (BLACK BOX)

Suppose $\ell(w_4) = \ell(w_1) + 2$. Then, $\exists$ exactly two $w, w' \in W$ s.t.

$$\begin{tikzcd}
& w \arrow[dr] & \\
w_1 \arrow[ur] \arrow[dr] & & w_4 \\
& w' \arrow[ur] &
\end{tikzcd}$$

$\square$

Example: If $W = W(I_2(4)) = \langle s, t \rangle$ dihedral



Claim: The matrix $E = E(\subseteq)$ satisfies

$$(*) \quad e(w_2,w_4)\, e(w_1,w_2) + e(w_3,w_4)\, e(w_1,w_3) = 0$$

whenever (w_1, w_2, w_3, w_4) form a square

Sketch:

Have $\begin{cases} w \to w' \iff w < w' \\ \varepsilon_K \text{ are determined by } \varepsilon|_{M(w.\lambda^0)}, \; \forall w \in W^{(\lambda)} \end{cases}$

From Fact

$$\begin{tikzcd} & w_2 \arrow[dr] & \\ w_1 \arrow[ur] \arrow[dr] & & w_4 \\ & w_3 \arrow[ur] & \end{tikzcd}$$

Wo apply that $\varepsilon^2 = 0$

$\square$

Thm (6.8) Given $(C^\circ_\bullet, \varepsilon_\bullet)$ a BGG-resol.

all $e(w,w') \neq 0$ when $\begin{cases} \ell(w) = k \\ \ell(w') = k+1 \end{cases}$.

and $w < w'$.

Pf: Downward induction on $k = \ell(w)$.

$k = m$, $(m-1)$ are clear:

$$M(w_0 \cdot \lambda^\circ) = M(\lambda) \xrightarrow{\varepsilon} L(\lambda) \qquad \text{non-zero}$$

$$\underbrace{\bigoplus_{\alpha\in\Delta} M(s_\alpha \cdot \lambda)}_{N(\lambda)} \longrightarrow M(\lambda) \qquad \underline{\text{non-zero}}$$

For the inductive hypothesis:

Lemma (6.7): $\alpha \in \Delta$, $\beta > 0$, $\alpha \neq \beta$. Then

$\exists$ diagram (L) $\Rightarrow$ $\exists$ diagram (R)

and vice-versa:

$$\text{(L)} \quad \begin{tikzcd} & w' \\ s_\alpha w \arrow[ur, "\beta"] \arrow[dr, "\alpha"'] & \\ & w \end{tikzcd} \qquad\qquad \begin{tikzcd} w' \arrow[dr, "\alpha"] & \\ & s_\alpha w' \\ w \arrow[ur, "\gamma = s_\alpha\beta"'] & \end{tikzcd} \quad \text{(R)}$$

Pf: (L) means: $\begin{cases} w' = \underline{s_\beta s_\alpha w} \\ \ell(w') = \ell(w) \\ = \ell(s_\alpha w) + 1 \end{cases}$

$$\gamma = s_\alpha \beta \quad \Rightarrow \quad s_\gamma = s_\alpha s_\beta s_\alpha$$

$$\Rightarrow \quad \underline{s_\gamma w} = s_\alpha (s_\beta s_\alpha w) \overset{(L)}{=} \underline{s_\alpha w'}$$

It remains to show $\underline{\ell(s_\alpha w')} = \underline{\ell(w')} \underline{+1}$

$(L) \Rightarrow \ell(s_\beta w') < \ell(w')$

$\Rightarrow (w')^{-1} \beta < 0$

$\Leftrightarrow (w')^{-1} s_\alpha \gamma < 0$ as $\gamma = s_\alpha \beta$

$\Leftrightarrow (s_\alpha w') \gamma < 0$

$$\Rightarrow \ell(\underbrace{s_\gamma \underbrace{s_\alpha w'}_{s_\gamma w}}_{w}) < \ell(s_\alpha w')$$

$$\overset{(L)}{\Longleftrightarrow} \underline{\ell(w')} = \underline{\ell(w)} < \underline{\ell(s_\alpha w')}$$

$$\left(0 \to C_0 = M(w_0 \cdot \lambda) \to C_1 \to \cdots \to C_{m-1} \to C_m = M(\lambda) \to L(\lambda) \to 0 \right) \quad \square$$

Back to the Theorem :

$0 \neq E_k \,|\, M(w \cdot \lambda^\circ) \Rightarrow \exists \beta > 0$ s.t.

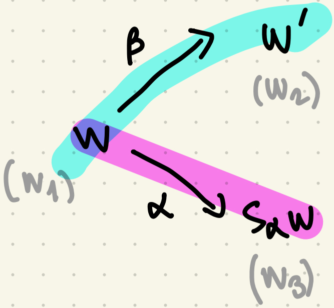
•) $w \xrightarrow{\beta} w'$

••) $\ell(w) = \ell(w') - 1$

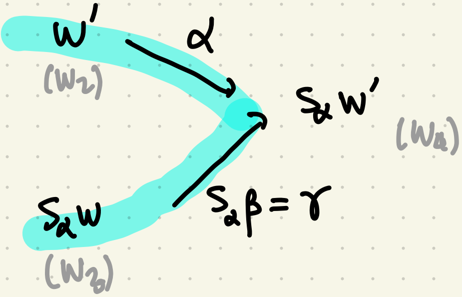
•••) $e(w, w') \neq 0$

$k \leq m-1 \Rightarrow \exists \alpha \in \Delta \quad w \xrightarrow{\alpha} s_\alpha w$

Case 1 : $\alpha \neq \beta$.



Lemma $\Longrightarrow$



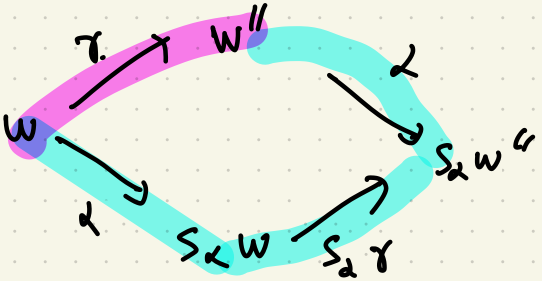
$$(*) \Rightarrow \underbrace{e(\overset{2}{w'}, \overset{4}{s_\alpha w'})}_{\neq 0 \text{ IH}} \underbrace{e(\overset{1}{w}, \overset{2}{w'})}_{\neq 0 \text{ ass.}} + \underbrace{e(\overset{3}{s_\alpha w}, \overset{4}{s_\alpha w'})}_{\neq 0 \text{ IH}} e(\overset{1}{w}, \overset{3}{s_\alpha w}) = 0$$

$\therefore \quad e(w, s_\alpha w) \neq 0$

Case 2 : $\beta = \alpha \Rightarrow (w \xrightarrow{\beta} w') = (w \xrightarrow{\alpha} s_\alpha w)$

$\Rightarrow e(w, s_\alpha w) = e(w, w') \neq 0.$

If $\gamma > 0$ s.t.



$\Rightarrow e(w, w'') \neq 0$

$\square$

Homological computations in $\mathcal{O}$

Let $M \in \mathcal{O}$, $\underline{P} = (P_\bullet, \partial_\bullet)$

$$\cdots \to P_n \to \cdots \to P_0 \to M \to 0$$

be a proj. resolution

Def: $\ell(\underline{P}) = n$ if $j > n \Rightarrow P_j = 0$ (length)

$\mathrm{pd}(M) = \inf\{\ell(\underline{P})\}$ (prj. dim)

$\mathrm{gd}(\mathcal{O}) = \sup\{\mathrm{pd}(M)\}$ (gl. dim.)

Notations: $E^n(M,N) = \mathrm{Ext}^n_{\mathcal{O}}(M,N)$,

$E^n(w,w') = E^n(M(w\cdot\lambda), M(w'\cdot\lambda))$, $E^0 = \mathrm{Hom}$

Main idea for the computations:

From SES

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[dr, dashed, bend right] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
& & D & &
\end{tikzcd}$$

Get LES

$$\begin{tikzcd}
\cdots \arrow[r] & E^n(C,D) \arrow[r] & E^n(B,D) \arrow[r] & E^n(A,D) \arrow[dll] & \\
& E^{n+1}(C,D) \arrow[r] & E^{n+1}(B,D) \arrow[r] & E^{n+1}(A,D) \arrow[dll] & \\
& E^{n+2}(C,D) \arrow[r] & E^{n+2}(B,D) \arrow[r] & E^{n+2}(A,D) \arrow[r] & \cdots
\end{tikzcd}$$

Lem: (A) $\operatorname{pd}(A) \leq \max(\operatorname{pd}(B), \operatorname{pd}(C) - 1)$

(B) $\operatorname{pd}(B) \leq \max(\operatorname{pd}(A), \operatorname{pd}(C))$

(C) $\operatorname{pd}(C) \leq \max(\operatorname{pd}(B), \operatorname{pd}(A) + 1)$

Sketch: $\operatorname{pd}(A) \leq n \Rightarrow E^k(B, D) \cong E^k(C, D) \quad k > n+1$

$\operatorname{pd}(B) \leq n \Rightarrow E^k(A, D) \cong E^{k+1}(C, D) \quad k > n$

$\square$

Thm 6.9 (a) $\operatorname{pd} M(w \cdot 0) = \ell(w)$

(b) $\operatorname{pd} L(w \cdot 0) = 2m - \ell(w)$

$\forall w \in W$. In particular $\text{gl. dim. } \mathcal{O}_0 = 2m$.

Pf: 0 max'l $W \cdot 0 \Rightarrow M(0)$ projective

$\Rightarrow \operatorname{pd}(M(0)) = 0$

Assume $\ell(w') < \ell(w)$

Claim: $\operatorname{pd}(M(w \cdot 0)) \leq \ell(w)$.

Pf: Have $0 \to \underset{\substack{\uparrow \\ \text{std filtration}}}{N} \to P(w \cdot 0) \to M(w \cdot 0) \to 0$

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{p-1} \subseteq N_p = N \subseteq P(w\cdot 0)$$

$$N_k / N_{k-1} \cong M(\mu_k)$$

$$0 \neq (P(w\cdot 0) : M(\mu_k)) = [M(\mu_k) : L(w\cdot 0)] \left(\begin{array}{l}\text{BGG} \\ \text{recipr.}\end{array}\right)$$

$$= [M(w'.0) : L(w\cdot 0)] \left(\begin{array}{l}\mu_k = w'.0 \\ \exists\, w'\end{array}\right)$$

$$(*) \qquad \Longleftrightarrow M(w\cdot 0) \hookrightarrow M(w'.0)$$

$$\Longleftrightarrow w \geqslant w'$$

$$\therefore \left\{ \begin{array}{l} 0 \to N \to P(w\cdot 0) \to M(w\cdot 0) \to 0 \\ 0 \to N_{k-1} \to N_k \to M(w'.0) \to 0 \end{array} \right.$$

$$\mathrm{pd}(N) \overset{(B)}{\leqslant} \underset{\forall k}{\mathrm{pd}(N_k)} \leqslant \mathrm{pd}(M(w'.0)) \quad \forall w' < w$$

$$\mathrm{pd}(M(w.0)) \underset{(C)}{\leqslant} \mathrm{pd}(N) + 1 \underset{(IH)}{\leqslant} \ell(w') + 1 = \ell(w)$$

$\blacksquare$

Claim $\mathrm{pd}(M(w\cdot 0)) \geqslant \ell(w)$.

Pf: Choose w' s.t. $\ell(w') = \ell(w) - 1$.

$$\Longrightarrow \quad \ell(w') \overset{(C)}{\underset{\text{I.H.}}{\leqslant}} \mathrm{pd}(N_k) \quad k+1,\ k+2, \ldots \quad (\exists\, k)$$

$$\mathrm{pd}(N_{k-1}) \overset{(A)}{\leqslant} \mathrm{pd}(N_k) \qquad (\forall k)$$

$$\therefore \quad \ell(w') \leqslant \mathrm{pd}(N) \overset{(A)}{\leqslant} \mathrm{pd}(M(w\cdot 0)) - 1$$

$$\Rightarrow \quad \ell(w) = \ell(w') + 1 \leqslant \mathrm{pd}(M(w\cdot 0)) \qquad \blacksquare$$

Rmk: Similar for $\mathrm{pd}(L(w\cdot 0))$ using

$$0 \to N(w\cdot 0) \to M(w\cdot 0) \to L(w\cdot 0) \to 0. \quad ●$$

6.10 Recap Homological computations

(4.2) $\dim E^0(M(\mu), M(\lambda)) \leqslant 1$

(5.1) $\exists\ M(\mu) \hookrightarrow M(\lambda) \iff \mu \uparrow \lambda$

(3.3) $\dim E^0(M(\mu), M(\nu)^\vee) = \delta_{\mu\nu}$

(3.3) $E^1(M(\mu), M(\lambda)^\vee) = 0$

(6.5) $E^1(M(\mu), M(\lambda)) \neq 0 \Rightarrow \mu \uparrow \lambda, \ \mu \neq \lambda$

Plan: Generalize last 2 results to

(a) all $n > 0$

(b) modules with std filtration

Thm 6.11 $\lambda \in \Lambda^+$, $w, w' \in W$

(a) $E^n(w', w) = 0 = E^n(M(w'.\lambda), L(w.\lambda))$

unless $w'.\lambda \uparrow w.\lambda$, $w' \neq w$ $(w \lneq w')$

(b) If $w'.\lambda \lneq w.\lambda$, then $\forall n > \ell(w') - \ell(w)$

$$E^n(w', w) = 0 = E^n(M(w'.\lambda), L(w.\lambda))$$

Pf (a) Assume $w \not< w'$.

- $0 \to N \to P(w'.\lambda) \to M(w'.\lambda) \to 0$
- $D = M(w.\lambda)$
- $n=1$ (6.5) true

$$\cdots \underbrace{E^n(N, w)}_{\text{I.H Claim S}} \to E^{n+1}(w', w) \to E^{n+1}(\underbrace{P(w'.\lambda), w}_{\text{proj.} = 0}) \cdots$$

s.t. filtr. factors x $\begin{cases} x < w' \\ w < x \end{cases}$

Claim S: N with std filtr, $E^n(M(\mu), D) = 0$

$\forall$ factors $\Rightarrow E^n(N, D) = 0$

(induction on length of verma flag)

Pf: $0 \to N_{p-1} \to N \to M(\mu) \to 0$

$$\rightsquigarrow \cdots \underbrace{E^n(\mu, D)}_{=0 \text{ assump.}} \to E^n(N, D) \to \underbrace{E^n(N_{p-1}, D)}_{IH = 0} \to \cdots$$

$\triangleleft$ □

Remark 1. For second part of (a):

$$0 \to N(w\cdot\lambda) \to M(w\cdot\lambda) \to L(w\cdot\lambda) \to 0$$

and the covariant version

$$\cdots \to E^n(w', N(w\cdot\lambda)) \to E^n(w', w) \to \cdots$$

2. (b) uses similar ideas!

Thm 6.12:

$$E^n(M(\mu), M(\lambda)^\vee) = 0 \quad \begin{cases} \forall n > 0 \\ \forall \lambda, \mu \in \mathfrak{g}^* \end{cases}$$

Pf: True for $n=1$ (3.3). Assume for n

$$0 \to N \to P(\mu) \to M(\mu) \to 0$$

$$D = M(\lambda)^\vee$$

$$\rightsquigarrow \cdots E^n(\underbrace{N, D}_{\substack{\text{Claim S.} \\ \text{IH - J}}}) \to \underset{\Rightarrow 0}{E^{n+1}(\mu, D)} \to E^{n+1}(\underbrace{P(\mu)}_{\text{proj} \Rightarrow 0}, D) \to \cdots$$

$\square$

$\square$

Thm 6.13 $M \in \mathcal{O}$. TFAE:

(a) M has std filtration

(b) $E^n(M, M(\lambda)^\vee) = 0 \quad \forall n > 0, \ \forall \lambda \in \mathfrak{h}^*$

(c) $E^1(M, M(\lambda)^\vee) = 0 \quad \forall \lambda \in \mathfrak{h}^*$

Pf: (c) $\Rightarrow$ (a): Induction (JH) length of M.

(1) Choose λ_0 minimal s.t.

$$E^0(M, L(\lambda_0)) \neq 0$$

(2) $E^0(M, L(\mu)) = 0$ if $\mu < \lambda_0$

Claim: $E^1(M, L(\mu)) = 0.$

Pf: Use

$$\begin{cases} 0 \to L(\mu)^\vee \to M(\mu)^\vee \to \overset{\overset{Q}{\shortparallel}}{M(\mu)^\vee / L(\mu)} \to 0 \\ D = M \end{cases}$$

$$\cdots \ \underbrace{E^0(D, Q)}_{(*)} \to E^1(D, L(\mu)) \to \underbrace{E^1(D, M(\mu)^\vee)}_{(c) \Rightarrow 0} \cdots$$

(*) $\mu < \lambda_0 \overset{(1)}{\Rightarrow} E^0(M, L(\nu)) = 0, \ \forall L(\nu) \leq Q$ a simple submodule ◁

(3) From $\left\{ \begin{array}{l} 0 \to N(\lambda_0) \to M(\lambda_0) \to L(\lambda_0) \to 0 \\ D = M \end{array} \right.$

$$\cdots\ \underbrace{E^0(D, N(\lambda_0))}_{(2) \Rightarrow 0} \to E^0(D, M(\lambda_0)) \to E^0(D, L(\lambda_0))$$

$$\to \underbrace{E^1(D, N(\lambda_0))}_{(2) \Rightarrow 0} \to \cdots$$

$$\begin{tikzcd}
\therefore\ M \arrow[rr, two heads] \arrow[dr] & & L(\lambda_0) \\
& M(\lambda_0) \arrow[ur] &
\end{tikzcd}$$

$E^1(M, M(\lambda)^\vee) \neq 0$

(4) $0 \to N \to M \twoheadrightarrow M(\lambda_0) \to 0$

Claim: $E^1(N, M(\mu)^\vee) = 0 \quad \forall \mu \in \mathfrak{h}^*$

Pf: Use $D = M(\mu)^\vee$ and

$$\cdots\ \underbrace{E^1(M, D)}_{(c) \Rightarrow 0} \to E^1(N, D) \to E^2(\underbrace{M(\lambda_0), D}_{6.12 \Rightarrow 0}) \to \cdots$$

◁

(5) (IH) $\Rightarrow$ N has std filtration

(4) $\Rightarrow$ M " " "

□