Highest weight modules - Part II

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Ghent University – Kleine Seminar

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5.1: The BGG Theorem

Let $\lambda, \mu \in \mathfrak{h}^*$

Definition: $\mu \uparrow \lambda$ (μ is strongly linked to λ)

$\mu = \lambda$ or $\exists \alpha_1, \ldots, \alpha_m \in \Phi^+$:
$\mu = (s_{\alpha_1} \ldots s_{\alpha_m}) \cdot \lambda < (s_{\alpha_2} \ldots s_{\alpha_m}) \cdot \lambda < \cdots < s_{\alpha_m} \cdot \lambda < \lambda$

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$\Leftrightarrow \langle (s_{\alpha_{i+1}} \ldots s_{\alpha_m}) \cdot \lambda + \rho, \alpha_i^\vee \rangle \in \mathbb{Z}^{>0}$ for all $i = 1, \ldots, m$

$$\underline{s_{\alpha_i} \cdots s_{\alpha_m}} \cdot \lambda = s_{\alpha_i} \cdot (s_{\alpha_{i+1}} \cdots s_{\alpha_m} \cdot \lambda)$$
$$= \underline{s_{\alpha_{i+1}} \cdots s_{\alpha_m} \cdot \lambda} - \underline{\langle s_{\alpha_{i+1}} \cdots s_{\alpha_m} \cdot \lambda + \rho, \alpha_i^\vee \rangle \alpha_i}$$

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$[M : L(\mu)]$ = multiplicity of $L(\mu)$ as a composition factor of M

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$[M : L(\mu)] =$ multiplicity of $L(\mu)$ as a composition factor of M

Theorem 4.6 [Verma]:

For any $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Phi^+$: $s_\alpha \cdot \lambda \leq \lambda \Rightarrow M(s_\alpha \cdot \lambda) \hookrightarrow M(\lambda)$

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$$\Rightarrow \quad \mu = (s_{\alpha_1} \dots s_{\alpha_m}) \cdot \lambda < (s_{\alpha_2} \dots s_{\alpha_m}) \cdot \lambda < \dots < s_{\alpha_m} \cdot \lambda < \lambda$$

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Iterating Theorem 4.6:
$M(\mu) \hookrightarrow M((s_{\alpha_2} \dots s_{\alpha_m}) \cdot \lambda) \hookrightarrow \cdots \hookrightarrow M(s_{\alpha_m} \cdot \lambda) \hookrightarrow M(\lambda)$
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Corollary

For any $\lambda, \mu \in \mathfrak{h}^*$: $\quad [M(\lambda) : L(\mu)] \neq 0 \Leftrightarrow M(\mu) \hookrightarrow M(\lambda)$

But $[M(\lambda) : L(\mu)] = m \not\Rightarrow m$ different embeddings

$\Leftarrow$

$\Rightarrow$

$\forall \alpha \in \Phi^+ : \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{<0}$

$\forall \alpha \in \Phi : \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}$

Proposition 4.3:

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Notation

- $\Phi_{[\lambda]} = \{\alpha \in \Phi : \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}$ is a root system
- $\Phi^+_{[\lambda]} := \Phi_{[\lambda]} \cap \Phi^+$ is a positive system in $\Phi_{[\lambda]}$
- $\Delta_{[\lambda]} :=$ simple system in $\Phi_{[\lambda]}$
- $W_{[\lambda]} = \{w \in W : w \cdot \lambda - \lambda \in \Lambda^r\}$ is its Weyl group, generated by $\{s_\alpha : \alpha \in \Delta_{[\lambda]}\}$

5.1: The BGG Theorem

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Proof: By induction on $\ell(w)$

- $\ell(w) = 0$: Trivial
- Write s_i for the reflection w.r.t. simple root $\alpha_i \in \Delta_{[\lambda]}$.
  Let $w = s_n \dots s_1$ be a reduced expression. Let $w' := s_{n-1} \dots s_1$. $\ell(w') < \ell(w)$

$\overset{IH}{\Longrightarrow} M(w' \cdot \lambda) \hookrightarrow M(\lambda)$.

TBP: $M(w \cdot \lambda) \hookrightarrow M(w' \cdot \lambda)$

By BGG: TBP: $w \cdot \lambda \uparrow w' \cdot \lambda$, so equiv. $\langle w' \cdot \lambda + \rho, \alpha_n^\vee \rangle \in \mathbb{Z}^{\geq 0}$.

$\langle w' \cdot \lambda + \rho, \alpha_n^\vee \rangle = \langle w'(\lambda + \rho), \alpha_n^\vee \rangle = \langle \lambda + \rho, w'^{-1}(\alpha_n^\vee) \rangle$

$= \langle \lambda + \rho, (w'^{-1}(\alpha_n))^\vee \rangle$

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For dominant weights λ:
$\forall w \in W_{[\lambda]} : M(w \cdot \lambda) \hookrightarrow M(\lambda)$ and hence $[M(\lambda) : L(w \cdot \lambda)] \neq 0$

Proof (cntd):

$w'^{-1}(\alpha_n) \in \Phi^+_{[\lambda]} = \Phi^+ \cap \Phi_{[\lambda]}$

$\hookrightarrow w'^{-1}(\alpha_n) \in \Phi_{[\lambda]} \Rightarrow \langle \lambda+\rho, (w'^{-1}(\alpha_n))^\vee \rangle \in \mathbb{Z}$

$\hookrightarrow w'^{-1}(\alpha_n) \in \Phi^+ \underset{\lambda \text{ dominant}}{\Rightarrow} \langle \lambda+\rho, (w^{-1}(\alpha_n))^\vee \rangle \notin \mathbb{Z}^{<0}$

$\Rightarrow \in \mathbb{Z}^{\geq 0}$

Exercise 5.1:

Suppose that $[M(\lambda) : L(\mu)] \in \{0, 1\}$ for all $\mu \in \mathfrak{h}^*$.
Show that $N(\lambda) = \bigoplus_{\mu:[M(\lambda):L(\mu)]=1} M(\mu)$

$$\sum_{\mu} M(\mu) \subseteq N(\lambda)$$

Also: $N(\lambda)$ contains all submodules of $M(\lambda)$, except $L(\lambda)$

So if $\sum_{\mu} M(\mu) \neq N(\lambda)$

$\Rightarrow \exists \gamma : [M(\lambda) : L(\gamma)] \neq 0 \wedge L(\gamma) \subseteq N(\lambda) \wedge L(\gamma) \not\subseteq \sum_{\mu} M(\mu)$

$\Downarrow$

$M(\gamma) \hookrightarrow M(\lambda) \quad \Rightarrow [M(\lambda) : L(\gamma)] \geq 2 \quad ↯$

5.2: The Bruhat ordering

Definition: Chevalley-Bruhat ordering on W: $w' \leq w$ if

$\exists \alpha_1, \dots, \alpha_m \in \Phi^+$: $w = s_{\alpha_m} \dots s_{\alpha_1} w'$ and
$\ell(w') < \ell(s_{\alpha_1} w') < \cdots < \ell(s_{\alpha_{m-1}} \dots s_{\alpha_1} w') < \ell(w)$

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Lemma 5.2:

For regular, antidominant, integral weights λ:
$\forall w, w' \in W$: $w' \cdot \lambda < w \cdot \lambda \Leftrightarrow w' < w$

The linkage class $\{w \cdot \lambda : w \in W\}$ of a regular, integral weight is indexed by its lowest weight, which is regular, integral and antidominant.

Regular: $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0, \quad \forall \alpha \in \Phi$

Antidominant: $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}, \quad \forall \alpha \in \Phi^+$

Lemma 5.2:

For regular, antidominant, integral weights λ:
$\forall w, w' \in W$: $w' \cdot \lambda < w \cdot \lambda \Leftrightarrow w' < w$

Proof:

First consider $w' = s_\alpha w$, $\alpha \in \Phi^+$

$$s_\alpha \cdot (w \cdot \lambda) = w' \cdot \lambda < w \cdot \lambda$$

$$\Updownarrow$$

$$\langle w \cdot \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^{>0}$$

$$\| $$

$$\langle \lambda + \rho, (w^{-1}(\alpha))^\vee \rangle$$

$$\Updownarrow \ \lambda \text{ regular, antidom, integr.}$$

$$w^{-1}(\alpha) \in \Phi^- \iff \underbrace{(w^{-1} s_\alpha)}_{= w'^{-1}}(\alpha) \in \Phi^+$$

$$w'^{-1}(\alpha)$$

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$\forall w, w' \in W$: $w' \cdot \lambda < w \cdot \lambda \Leftrightarrow w' < w$

Proof (cntd):

$w' \cdot \lambda < w \cdot \lambda \Leftrightarrow w'^{-1}(\alpha) \in \Phi^+ \overset{0.3}{\Leftrightarrow} \ell(w) = \ell(s_\alpha w') > \ell(w')$

$\Leftrightarrow w > w'$.

For $w' = s_\alpha w$

Corollary 5.2:

For regular, antidominant, integral weights λ:
$\forall w, w' \in W$: $[M(w\cdot\lambda) : L(w'\cdot\lambda)] \neq 0 \Leftrightarrow w' \leq w$

Proof:

$\boxed{\Rightarrow}$ $[M(w\cdot\lambda) : L(w'\cdot\lambda)] \neq 0 \overset{BGG}{\Longrightarrow} w'\cdot\lambda \uparrow w\cdot\lambda \Rightarrow w'\cdot\lambda \leq w\cdot\lambda$

$\overset{\text{Lemma 5.2}}{\Longrightarrow} w' \leq w.$

$\boxed{\Leftarrow}$ $w' \leq w \Rightarrow \exists \alpha_1, \dots, \alpha_n \in \Phi^+ : w' < s_{\alpha_1} w' < \dots < s_{\alpha_n} \cdots s_{\alpha_1} w' = w$

$\overset{\text{Lemma 5.2}}{\Longrightarrow} \underbrace{w'\cdot\lambda}_{= s_{\alpha_1}\cdots s_{\alpha_n} w\cdot\lambda} < \underbrace{s_{\alpha_1} w'\cdot\lambda}_{= s_{\alpha_2}\cdots s_{\alpha_n} w\cdot\lambda} < \dots < \underbrace{s_{\alpha_{n-1}} \cdots s_{\alpha_1} w'\cdot\lambda}_{= s_{\alpha_n} w\cdot\lambda} < w\cdot\lambda$

$\Rightarrow w'\cdot\lambda \uparrow w\cdot\lambda \overset{BGG}{\Longrightarrow} [M(w\cdot\lambda) : L(w'\cdot\lambda)] \neq 0.$

5.3: The Jantzen filtration

Definition: Contravariant form $(\cdot,\cdot)_M$ on $U(\mathfrak{g})$-module M

Symmetric bilinear form:
$(u \cdot v, v') = (v, \tau(u) \cdot v'), \quad \forall v, v' \in M, u \in U(\mathfrak{g})$
Here $\tau : U(\mathfrak{g}) \to U(\mathfrak{g}) : x_\alpha \mapsto y_\alpha, y_\alpha \mapsto x_\alpha, h \mapsto h$

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Theorem 5.3 [Jantzen]:

For any $\lambda \in \mathfrak{h}^*$, $M(\lambda)$ has a filtration
$M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \cdots \supset M(\lambda)^n \supset M(\lambda)^{n+1} = \{0\}$
such that $\forall i = 0, \ldots, n$

0) $M(\lambda)^i$ is nontrivial submodule of $M(\lambda)$

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0) $M(\lambda)^i$ is nontrivial submodule of $M(\lambda)$

1) $\underbrace{M(\lambda)^i / M(\lambda)^{i+1}}_{\text{the } i\text{-th filtration layer}}$ has a non-degenerate contravariant form

$\hookrightarrow (v, \cdot) = 0 \Rightarrow v = 0$

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0) $M(\lambda)^i$ is nontrivial submodule of $M(\lambda)$
1) $M(\lambda)^i / M(\lambda)^{i+1}$ has a non-degenerate contravariant form
2) $M(\lambda)^1 = N(\lambda)$ $\qquad \Phi^+_\lambda \subseteq \Phi^+ \cap \Phi_{[\lambda]} = \Phi^+_{[\lambda]}$
3) Jantzen sum formula: $\displaystyle\sum_{i=1}^{n} \mathrm{ch}\ M(\lambda)^i = \sum_{\alpha \in \Phi^+_\lambda} \mathrm{ch}\ M(s_\alpha \cdot \lambda)$

Here $\Phi^+_\lambda = \{\alpha \in \Phi^+ : s_\alpha \cdot \lambda < \lambda\} = \{\alpha \in \Phi^+ : \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^{>0}\}$

5.3: The Jantzen filtration

Exercise 5.3.1:

Let $L(\widetilde{\lambda})$ be the unique simple submodule of $M(\lambda)$.
Show that $[M(\lambda) : L(\widetilde{\lambda})] = 1 \Rightarrow n = |\Phi_{\lambda}^{+}|$

In Chapter 7, we will prove that $[M(\lambda) : L(\widetilde{\lambda})] = 1$ for all $\lambda \in \mathfrak{h}^*$.

This was already proven in Theorem 4.10 for integral weights λ.

Exercise 5.3.2:

Let λ be regular, antidominant and integral. Let $w \in W$.
Show that in the filtration of $M(w \cdot \lambda)$: $n = \ell(w)$.

$$n = |\Phi^+_{w\cdot\lambda}| = |\{\alpha \in \Phi^+ : s_\alpha w \cdot \lambda < w \cdot \lambda\}|$$
$$= |\{\alpha \in \Phi^+ : \underbrace{\langle w\cdot\lambda + \rho, \alpha^\vee \rangle}_{\langle \lambda+\rho, (w^{-1}(\alpha))^\vee\rangle} \in \mathbb{Z}^{>0}\}|$$

Since λ is regular, antidom., integral:

$$\langle \lambda + \rho, (w^{-1}(\alpha))^\vee \rangle \in \mathbb{Z}^{>0} \iff w^{-1}(\alpha) \in \Phi^-$$

$$n = |\{\alpha \in \Phi^+ : w^{-1}(\alpha) \in \Phi^-\}| = \ell(w^{-1}) = \ell(w).$$

Theorem [Jantzen Conjecture]:

For any $\lambda, \mu \in \mathfrak{h}^*$ with $\mu \uparrow \lambda$:

Set $r = |\Phi^+_\lambda| - |\Phi^+_\mu|$

- $M(\mu) \hookrightarrow M(\lambda)^i, \quad \forall i \leq r$
- $M(\mu) \cap M(\lambda)^i = M(\mu)^{i-r}, \quad \forall i \geq r$

Proof: Requires Kazhdan-Lusztig theory (Chapter 8).

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- $\forall w \in W$: $L(\lambda)$ is the unique simple submodule of $M(w \cdot \lambda)$,
$[M(w \cdot \lambda) : L(\lambda)] = 1, \quad [M(w \cdot \lambda) : L(w \cdot \lambda)] = 1$

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  $[M(w \cdot \lambda) : L(\lambda)] = 1, \quad [M(w \cdot \lambda) : L(w \cdot \lambda)] = 1$
- $M(\lambda) = L(\lambda)$

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- $\forall w \in W$: $L(\lambda)$ is the unique simple submodule of $M(w \cdot \lambda)$, $[M(w \cdot \lambda) : L(\lambda)] = 1, \quad [M(w \cdot \lambda) : L(w \cdot \lambda)] = 1$
- $M(\lambda) = L(\lambda)$
- $[M(s_\alpha \cdot \lambda) : L(w' \cdot \lambda)] = \begin{cases} 1 & \text{for } w' \in \{1, s_\alpha\}, \\ 0 & \text{else} \end{cases}$

  $\mathrm{ch}\ M(s_\alpha \cdot \lambda) = \mathrm{ch}\ L(s_\alpha \cdot \lambda) + \mathrm{ch}\ L(\lambda)$
  Similarly for $M(s_\beta \cdot \lambda)$

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  Similarly for $M(s_\beta \cdot \lambda)$

Now let's consider the composition factors of $M(s_\alpha s_\beta \cdot \lambda)$

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- We already know:
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  $$\Phi^+_{s_\alpha s_\beta \cdot \lambda} = \{\gamma \in \Phi^+ : s_\gamma s_\alpha s_\beta \cdot \lambda < s_\alpha s_\beta \cdot \lambda\} = \{\gamma \in \Phi^+ : s_\gamma s_\alpha s_\beta < s_\alpha s_\beta\} = \{\alpha, \beta\}$$

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- By Jantzen sum formula:
$$\begin{aligned}\sum_{i=1}^{n_{s_\alpha s_\beta \cdot \lambda}} \mathrm{ch}\ M(\lambda)^i &= \mathrm{ch}\ M(s_\alpha \cdot \lambda) + \mathrm{ch}\ M(s_\beta \cdot \lambda) \\ &= \mathrm{ch}\ \underbrace{L(s_\alpha \cdot \lambda)} + \mathrm{ch}\ L(s_\beta \cdot \lambda) + 2\,\mathrm{ch}\ \underbrace{L(\lambda)}\end{aligned}$$

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  $\Phi^+_{s_\alpha s_\beta \cdot \lambda} = \{\gamma \in \Phi^+ : s_\gamma s_\alpha s_\beta \cdot \lambda < s_\alpha s_\beta \cdot \lambda\} = \{\gamma \in \Phi^+ : s_\gamma s_\alpha s_\beta < s_\alpha s_\beta\} = \{\alpha, \beta\}$
- By Jantzen sum formula:
  $$\sum_{i=1}^{n_{s_\alpha s_\beta \cdot \lambda}} \mathrm{ch}\ M(\lambda)^i = \mathrm{ch}\ M(s_\alpha \cdot \lambda) + \mathrm{ch}\ M(s_\beta \cdot \lambda) = \mathrm{ch}\ L(s_\alpha \cdot \lambda) + \mathrm{ch}\ L(s_\beta \cdot \lambda) + 2\,\mathrm{ch}\ L(\lambda)$$
  $\Rightarrow L(s_\alpha \cdot \lambda)$ and $L(s_\beta \cdot \lambda)$ occur exactly once as composition factor of $M(\lambda)^1$ and hence of $M(\lambda)$

5.4: Application to $\mathfrak{sl}(3, \mathbb{C})$

- We already know:
  $[M(s_\alpha s_\beta \cdot \lambda) : L(\lambda)] = 1, \quad [M(s_\alpha s_\beta \cdot \lambda) : L(s_\alpha s_\beta \cdot \lambda)] = 1$
- By Lemma 5.2:
  $\Phi^+_{s_\alpha s_\beta \cdot \lambda} = \{\gamma \in \Phi^+ : s_\gamma s_\alpha s_\beta \cdot \lambda < s_\alpha s_\beta \cdot \lambda\} = \{\gamma \in \Phi^+ : s_\gamma s_\alpha s_\beta < s_\alpha s_\beta\} = \{\alpha, \beta\}$
- By Jantzen sum formula:

$$\sum_{i=1}^{n_{s_\alpha s_\beta \cdot \lambda}} \text{ch } M(\lambda)^i = \text{ch } M(s_\alpha \cdot \lambda) + \text{ch } M(s_\beta \cdot \lambda)$$

$s_\alpha s_\beta \cdot$

$$= \text{ch } L(s_\alpha \cdot \lambda) + \text{ch } L(s_\beta \cdot \lambda) + 2\,\text{ch } L(\lambda)$$

$\Rightarrow L(s_\alpha \cdot \lambda)$ and $L(s_\beta \cdot \lambda)$ occur exactly once as composition factor of $M(\lambda)^1$ and hence of $M(\lambda)$

$s_\alpha s_\beta \cdot$

$$\Rightarrow [M(s_\alpha s_\beta \cdot \lambda) : L(w \cdot \lambda)] = \begin{cases} 1 & \text{if } w \in \{1, s_\alpha, s_\beta, s_\alpha s_\beta\} \\ 0 & \text{else} \end{cases}$$

$s_\alpha s_\beta \cdot$

$\Rightarrow n_{s_\alpha s_\beta \cdot \lambda} = 2$ and $M(\lambda)^2 = L(\lambda)$

$$\text{ch} L(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e(w.\lambda)}{\sum_{w \in W} (-1)^{\ell(w)} e(0.\lambda)}.$$

Similarly for $M(s_\beta s_\alpha \cdot \lambda)$ and $M(w_0 \cdot \lambda)$

5.4: Application to $\mathfrak{sl}(3, \mathbb{C})$

Exercise 5.4:

Consider the simple block $\mathcal{O}_0$ for $\mathfrak{sl}(3, \mathbb{C})$. It has 6 simple modules $L(w \cdot (-2\rho))$, $w \in W$.

- For $w = 1$: $L(-2\rho) = M(-2\rho)$ and $\mathrm{ch}\ L(-2\rho) =$ partition function $\mathcal{P}$
  $\hookrightarrow (\mathcal{P} * e)(-2\rho)$
- For $w \neq 1$:
  - Compute $\mathrm{ch}\ L(w \cdot (-2\rho))$
  - Show that all weight spaces have dimension 1

$w_0 \cdot (-2\rho) = \underbrace{w_0(\rho)}_{=\rho} - \rho$

5.5: Proof of BGG theorem from Jantzen's theorem

Unofficial notation: For $\lambda \in \mathfrak{h}^*$:

$$N_\lambda = |\{\gamma \in \mathfrak{h}^* : \exists w \in W : \gamma = w \cdot \lambda \text{ and } \gamma < \lambda\}|$$

5.5: Proof of BGG theorem from Jantzen's theorem

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Theorem 5.1 [Bernstein-Gelfand-Gelfand]:

For any $\lambda, \mu \in \mathfrak{h}^*$: $\mu \uparrow \lambda \Leftrightarrow [M(\lambda) : L(\mu)] \neq 0$

Proof:

$\Leftarrow$ By induction on N_λ:

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- If $N_\lambda = 0$, then λ is minimal in its linkage class $\Rightarrow M(\lambda) = L(\lambda)$.
- For $N_\lambda > 0$:
  $[M(\lambda) : L(\mu)] > 0 \Rightarrow [M(\lambda)^1 : L(\mu)] > 0$ since $M(\lambda)^1 = N(\lambda)$

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  By Jantzen sum formula: $\exists \alpha \in \Phi_\lambda^+ : [M(s_\alpha \cdot \lambda) : L(\mu)] > 0$

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  $[M(\lambda) : L(\mu)] > 0 \Rightarrow [M(\lambda)^1 : L(\mu)] > 0$ since $M(\lambda)^1 = N(\lambda)$
  By Jantzen sum formula: $\exists \alpha \in \Phi_\lambda^+ : [M(s_\alpha \cdot \lambda) : L(\mu)] > 0$
  $s_\alpha \cdot \lambda < \lambda \Rightarrow N_{s_\alpha \cdot \lambda} < N_\lambda$
  By induction hypothesis: $\mu \uparrow s_\alpha \cdot \lambda$, and also $s_\alpha \cdot \lambda \uparrow \lambda$, so $\mu \uparrow \lambda$ $\square$

5.5: Proof of BGG theorem from Jantzen's theorem

Note: $w \cdot \lambda < \lambda \not\Rightarrow w \cdot \lambda \uparrow \lambda$

5.5: Proof of BGG theorem from Jantzen's theorem

Note: $w \cdot \lambda < \lambda \not\Rightarrow w \cdot \lambda \uparrow \lambda$

Example in $\mathfrak{sl}(4, \mathbb{C})$:
$\lambda = \varpi_1 - 2\varpi_2 - \varpi_3$, $w = s_2 s_3 s_2 s_1 s_2$
One can show: $\lambda + \rho = \alpha_1$, $w(\lambda + \rho) = -\alpha_3$
$\Rightarrow \lambda - w \cdot \lambda = \lambda + \rho - w(\lambda + \rho) = \alpha_1 + \alpha_3 > 0$
But Verma has shown:
$\nexists$ embedding of $M(w \cdot \lambda)$ in $M(\lambda) \overset{\text{Theorem 5.1}}{\Rightarrow} w \cdot \lambda$ is not strongly linked to λ

5.6: Some general theory of free modules over PIDs

Definitions:

- PID (principal ideal domain) A: commutative ring with 1, without zero divisors, in which every ideal is generated by a single element
- unit in A: invertible element
- prime element $p \in A$: $p \neq 0$, p is not a unit, $p|ab \Rightarrow p|a$ or $p|b$

Property:

A/pA is a field

5.6: Some general theory of free modules over PIDs

Let M be a free A-module of finite rank, with symmetric, non-degenerate bilinear form $(\cdot,\cdot)_M : M \times M \to A$ ↳ R

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- $D :=$ determinant of $(\cdot,\cdot)_M$, well-defined up to a unit, $D \neq 0$
  ↳ change of basis: $GL(r, A) \rightsquigarrow \det = \text{unit}$

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- Dual basis $\{e_1^*, \ldots, e_r^*\}$ is basis for M^*: $e_i^*(e_j) = \delta_{ij}$

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$\Rightarrow (e_i, f_j)_M = f_j^\vee(e_i) = d_j e_j^*(e_i) = d_j \delta_{ij}$, hence $D = \det((e_i, f_j)_M)_{ij} = \prod_{i=1}^r d_i$

5.6: Some general theory of free modules over PIDs

Let $p \in A$ be prime element

Definitions:

- For any $n \in \mathbb{Z}^{\geq 0}$: $M(n) := \{e \in M : (e, f)_M \in p^n A, \forall f \in M\} \subseteq M$
  Note: $M = M(0) \supseteq M(1) \supseteq M(2) \supseteq \dots$

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- For any $a \in A$: $v_p(a) := n \in \mathbb{Z}^{\geq 0}$ such that $p^n | a$ but $p^{n+1} \nmid a$
  Note $v_p(ab) = v_p(a) + v_p(b)$

$$v_p(ab) = n \Rightarrow p^n | ab \overset{p \text{ prime}}{\Longrightarrow} \exists j \in \{0, \dots, n\} : p^j | a \wedge p^{n-j} | b$$

$$p^{n+1} \nmid ab \Rightarrow p^{j+1} \nmid a \wedge p^{n-j+1} \nmid b \Rightarrow \begin{array}{l} v_p(a) = j \\ v_p(b) = n-j \end{array}$$

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  Note $v_p(ab) = v_p(a) + v_p(b)$
- $\overline{M} := M/pM$, considered as vector space over the field $\overline{A} := A/pA$
  Set $\overline{e} := e + pM$ for $e \in M$
- $\overline{M(n)} := M(n)/pM(n)$

Lemma 5.6.1:

$v_p(D) = \sum_{n=1}^{\infty} \dim(\overline{M(n)})$ and $\overline{M(n)} = 0$ for large enough n

Proof: $f = \sum_{j=1}^{n} a_j f_j \in M, \ a_j \in A$

$f \in M(n) \Leftrightarrow \forall e \in M: (e, f)_M \in p^n A \Leftrightarrow \forall i = 1, \dots, r: (e_i, f)_M \in p^n A$

$\Leftrightarrow \forall i = 1, \dots, r: \ v_p[\underbrace{(e_i, f)_M}_{\sum_{j=1}^{n} a_j \underbrace{(e_i, f_j)_M}_{d_i \delta_{ij}} = a_i d_i}] \geq n \qquad p^n | (e_i, f)_M$

$p^{n-n_i} | a_i$

$\Leftrightarrow \forall i: v_p(a_i) + \underbrace{v_p(d_i)}_{= n_i} \geq n \Leftrightarrow \forall i: v_p(a_i) \geq n - n_i.$

Spanning set for $M(n)$: $\{f_i : n \leq n_i\} \cup \{p^{n-n_i} f_i : n > n_i\}$

Basis for $\overline{M(n)}$: $\{\overline{f_i} : n \leq n_i\}$ (mod p), $\{0\}$

Lemma 5.6.1:

$v_p(D) = \sum_{n=1}^{\infty} \dim(\overline{M}(n))$ and $\overline{M}(n) = 0$ for large enough n

Proof (cntd):

$\dim(\overline{M(n)}) = |\{i = 1, \ldots, r : n \leq n_i\}|$

If $n > \max(n_1, \ldots, n_r) \Rightarrow \overline{M(n)} = 0$

$$\sum_{n=1}^{\infty} \dim(\overline{M(n)}) = \sum_{n=1}^{\infty} |\{i = 1, \ldots, r : n \leq n_i\}| = \sum_{i=1}^{r} \underbrace{n_i}_{= v_p(d_i)}$$

$$= v_p\left(\prod_{i=1}^{r} d_i\right) = v_p(D).$$

Lemma 5.6.2: For any $n \in \mathbb{Z}^{\geq 0}$

The modified symmetric bilinear form on $M(n)$ given by
$$(e, f)_n := p^{-n}(e, f)_M$$
induces a non-degenerate symmetric bilinear form on $\overline{M(n)}/\overline{M(n+1)}$.

Proof: $(\cdot,\cdot)_{\bar{n}} : \overline{M(n)} \times \overline{M(n)} \to \bar{A} : (\bar{e}, \bar{f})_{\bar{n}} = p^{-n}(e, f)_M \mod pA$

- Well-defined:

$\bar{e} = 0 \Rightarrow e \in pM$, write $e = pe'$

Let $f \in M(n)$:

$$(\bar{e}, \bar{f})_{\bar{n}} = p^{-n}(e, f)_M = p^{-n+1}(e', f)_M \underset{f \in M(n)}{\in} p^{-n+1} \cdot p^n A = pA = 0 \mod pA$$
$$= 0_{\bar{A}}.$$

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Proof (cntd):

$$\text{Radical} \overset{!}{=} \overline{M(n+1)}$$

$$\bar{f}_j \in \mathrm{Rad} \iff \forall i: \underbrace{(\bar{e}_i, \bar{f}_j)_{\bar{n}}}_{\substack{\parallel \\ p^{-n}(e_i, f_j)_n = p^{-n} d_j \delta_{ij}}} = 0 \mod pA \iff p^{-n} d_j = 0 \mod pA$$

$$\iff p^{n+1} \mid d_j \iff \underbrace{v_p(d_j)}_{\substack{\parallel \\ n_j}} \geq n+1$$

$$\text{Basis for Rad} = \{\bar{f}_j : n_j \geq n+1\} = \text{basis for } \overline{M(n+1)}.$$

5.7: Proof of Jantzen's theorem from the key lemma

Theorems 3.15' (3.15 and 4.8 combined): For any $\lambda \in \mathfrak{h}^*$

$\exists$ unique (up to scalar multiples) contravariant form $(\cdot,\cdot)_{M(\lambda)}$ on $M(\lambda)$, which is non-degenerate $\Leftrightarrow$ $M(\lambda)$ is simple $\Leftrightarrow$ λ is antidominant

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Let $\lambda \in \mathfrak{h}^*$ be fixed. Let T be any indeterminate.

Notations:

- $A = \mathbb{C}[T]$ (PID with prime element T) $\qquad K = \mathbb{C}(T)$ (field)

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  $\mathfrak{g}_A = A \otimes_{\mathbb{C}} \mathfrak{g} \qquad \mathfrak{g}_K = K \otimes_{\mathbb{C}} \mathfrak{g}$
  $U(\mathfrak{g}_A) \cong A \otimes_{\mathbb{C}} U(\mathfrak{g}) \qquad U(\mathfrak{g}_K) = K \otimes_{\mathbb{C}} U(\mathfrak{g})$

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- $\lambda_T := \lambda + T\rho \in \mathfrak{h}_K^*$
- $M(\lambda_T) = U(\mathfrak{g}_K)$-Verma module of highest weight λ_T
  $M(\lambda_T)^A =$ restriction to A

5.7: Proof of Jantzen's theorem from the key lemma

Properties: $\mathbb{Z}^{\geq 0}\alpha_1 \oplus \dots \oplus \mathbb{Z}^{\geq 0}\alpha_\ell$

- $\forall \nu \in \Gamma$: the weight space $M(\lambda_T)^A_{\lambda_T - \nu}$ is a free A-module of finite rank
  $\Rightarrow$ so is $\sum_{\nu \in \Gamma} M(\lambda_T)^A_{\lambda_T - \nu}$

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- $\forall \nu \in \Gamma$: the weight space $M(\lambda_T)^A_{\lambda_T - \nu}$ is a free A-module of finite rank $\Rightarrow$ so is $\sum_{\nu \in \Gamma} M(\lambda_T)^A_{\lambda_T - \nu}$
- λ_T is antidominant,
  since $\forall \alpha \in \Phi : \langle \rho, \alpha^\vee \rangle \neq 0$, so $\langle \lambda_T + \rho, \alpha^\vee \rangle = \underbrace{\langle \lambda + \rho, \alpha^\vee \rangle} + T \underbrace{\langle \rho, \alpha^\vee \rangle} \notin \mathbb{Z}$

5.7: Proof of Jantzen's theorem from the key lemma

Properties:

- $\forall \nu \in \Gamma$: the weight space $M(\lambda_T)^A_{\lambda_T - \nu}$ is a free A-module of finite rank $\Rightarrow$ so is $\sum_{\nu \in \Gamma} M(\lambda_T)^A_{\lambda_T - \nu}$
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5.7: Proof of Jantzen's theorem from the key lemma

Properties:

- $\forall \nu \in \Gamma$: the weight space $M(\lambda_T)^A_{\lambda_T-\nu}$ is a free A-module of finite rank $\Rightarrow$ so is $\sum_{\nu\in\Gamma} M(\lambda_T)^A_{\lambda_T-\nu}$
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  $\xRightarrow{\text{Theorem } 3.15'} (\cdot,\cdot)_{M(\lambda_T)}$ is non-degenerate + contravariant
  Its restriction to $\sum_{\nu\in\Gamma} M(\lambda_T)^A_{\lambda_T-\nu}$ remains non-degenerate and contravariant

5.7: Proof of Jantzen's theorem from the key lemma

Properties:

- $\forall \nu \in \Gamma$: the weight space $M(\lambda_T)^A_{\lambda_T-\nu}$ is a free A-module of finite rank $\Rightarrow$ so is $\sum_{\nu\in\Gamma} M(\lambda_T)^A_{\lambda_T-\nu}$
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$\Rightarrow \forall n \in \mathbb{Z}^{\geq 0}$ we may consider $\sum_{\nu\in\Gamma} M(\lambda_T)^A_{\lambda_T-\nu}(n)$

5.7: Proof of Jantzen's theorem from the key lemma

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$\Rightarrow \forall n \in \mathbb{Z}^{\geq 0}$ we may consider $\sum_{\nu\in\Gamma} M(\lambda_T)^A_{\lambda_T-\nu}(n)$

Definition:

$$M(\lambda_T)^{A(i)} := \sum_{\nu\in\Gamma} M(\lambda_T)^A_{\lambda_T-\nu}(i) = \Big\{ e \in \sum_{\nu\in\Gamma} M(\lambda_T)^A_{\lambda_T-\nu} : \forall f \in \ldots : (e,f)_{M(\lambda_T)} \in T^i A \Big\}$$

5.7: Proof of Jantzen's theorem from the key lemma

Properties:

- $\forall \nu \in \Gamma$: the weight space $M(\lambda_T)^A_{\lambda_T - \nu}$ is a free A-module of finite rank
  $\Rightarrow$ so is $\sum_{\nu\in\Gamma} M(\lambda_T)^A_{\lambda_T-\nu}$
- λ_T is antidominant,
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  $\overset{\text{Theorem } 3.15'}{\Longrightarrow} (\cdot,\cdot)_{M(\lambda_T)}$ is non-degenerate
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$\Rightarrow \forall n \in \mathbb{Z}^{\geq 0}$ we may consider $\sum_{\nu\in\Gamma} M(\lambda_T)^A_{\lambda_T-\nu}(n)$

Definition:

$$M(\lambda_T)^{A^{(i)}} := \sum_{\nu\in\Gamma} M(\lambda_T)^A_{\lambda_T-\nu}(i)$$
$$M(\lambda)^i := M(\lambda_T)^{A^{(i)}}|_{T=0}$$

5.7: Proof of Jantzen's theorem from the key lemma

Theorem 5.3 [Jantzen]:

$M(\lambda)$ has a filtration
$M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \cdots \supset M(\lambda)^n \supset M(\lambda)^{n+1} = \{0\}$
such that $\forall i = 0, \ldots, n$

0) $M(\lambda)^i$ is nontrivial submodule of $M(\lambda)$
1) $M(\lambda)_i := M(\lambda)^i / M(\lambda)^{i+1}$ has a non-degenerate contravariant form

Proof:

5.7: Proof of Jantzen's theorem from the key lemma

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Proof:
$M(\lambda_T)^A$ has a filtration $M(\lambda_T)^A = M(\lambda_T)^{A^{(0)}} \supset M(\lambda_T)^{A^{(1)}} \supset M(\lambda_T)^{A^{(2)}} \supset \ldots$

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Proof:
$M(\lambda_T)^A$ has a filtration $M(\lambda_T)^A = M(\lambda_T)^{A(0)} \supset M(\lambda_T)^{A(1)} \supset M(\lambda_T)^{A(2)} \supset \dots$
$\forall i \in \mathbb{Z}^{\geq 0}$:

0) $M(\lambda_T)^{A(i)}$ is a $U(\mathfrak{g}_A)$-submodule of $M(\lambda_T)^A$

5.7: Proof of Jantzen's theorem from the key lemma

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$M(\lambda_T)^A$ has a filtration $M(\lambda_T)^A = M(\lambda_T)^{A(0)} \supset M(\lambda_T)^{A(1)} \supset M(\lambda_T)^{A(2)} \supset \ldots$
$\forall i \in \mathbb{Z}^{\geq 0}$:

0) $M(\lambda_T)^{A^{(i)}}$ is a $U(\mathfrak{g}_A)$-submodule of $M(\lambda_T)^A$
$\Rightarrow M(\lambda)^i$ is a submodule of $M(\lambda)$

1) By Lemma 5.6.2, the induced bilinear form on $\overline{M(\lambda_T)^{A^{(i)}}} / \overline{M(\lambda_T)^{A^{(i+1)}}}$ is non-degenerate and contravariant

5.7: Proof of Jantzen's theorem from the key lemma

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1) $M(\lambda)_i := M(\lambda)^i / M(\lambda)^{i+1}$ has a non-degenerate contravariant form

Proof:
$M(\lambda_T)^A$ has a filtration $M(\lambda_T)^A = M(\lambda_T)^{A(0)} \supset M(\lambda_T)^{A(1)} \supset M(\lambda_T)^{A(2)} \supset \dots$
$\forall i \in \mathbb{Z}^{\geq 0}$:

0) $M(\lambda_T)^{A^{(i)}}$ is a $U(\mathfrak{g}_A)$-submodule of $M(\lambda_T)^A$
$\Rightarrow M(\lambda)^i$ is a submodule of $M(\lambda)$

1) By Lemma 5.6.2, the induced bilinear form on $\overline{M(\lambda_T)^{A^{(i)}}} / \overline{M(\lambda_T)^{A^{(i+1)}}}$ is non-degenerate and contravariant
$M(\lambda)^i$
$\Rightarrow$ Under $T = 0$ these yield non-degenerate contravariant forms on $M(\lambda)^i / M(\lambda)^{i+1}$

5.7: Proof of Jantzen's theorem from the key lemma

Theorem 5.3 [Jantzen]:

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such that $\forall i = 0, \ldots, n$

0) $M(\lambda)^i$ is nontrivial submodule of $M(\lambda)$
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Proof (cntd):

We have $M(\lambda)^{(n+1)} = \{0\}$ for n big enough, since

5.7: Proof of Jantzen's theorem from the key lemma

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1) $M(\lambda)^i / M(\lambda)^{i+1}$ has a non-degenerate contravariant form

Proof (cntd):
We have $M(\lambda)^{(n+1)} = \{0\}$ for n big enough, since

- $\forall \nu \in \Gamma : \exists n_\nu \in \mathbb{Z}^{\geq 0} : M(\lambda_T)^A_{\lambda_T - \nu}(n_\nu) = \{0\}$
- Only finitely many $\nu \in \Gamma$ are such that $\lim_{T \to 0}(\lambda_T - \nu) = \lambda - \nu$ is W-linked to λ and hence lead to non-trivial weight spaces in the limit $T = 0$ $\square$

5.7: Proof of Jantzen's theorem from the key lemma

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$M(\lambda)$ has a filtration
$M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \cdots \supset M(\lambda)^n \supset M(\lambda)^{n+1} = \{0\}$
such that

2) $M(\lambda)^1 = N(\lambda)$

Proof:

5.7: Proof of Jantzen's theorem from the key lemma

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such that

2) $M(\lambda)^1 = N(\lambda)$

Proof:
$M(\lambda)/M(\lambda)^1$ is a highest weight module with non-degenerate contravariant form

$\overset{\text{Theorem 3.15}}{\Longrightarrow}$ $M(\lambda)/M(\lambda)^1$ is simple

$\Longrightarrow$ $M(\lambda)^1$ is the maximal submodule of $M(\lambda)$, i.e. $N(\lambda)$ □

5.7: Proof of Jantzen's theorem from the key lemma

$\lambda_T = \lambda + Te$ $\quad K = \mathbb{C}(T)$ $\quad A = \mathbb{C}[T]$

$M(\lambda_T)^A_{\lambda_T - \nu}$ $\quad \nu \in \Gamma$

Definitions:

- $D_\nu(\lambda_T) := \det\left[(\cdot,\cdot)_{M(\lambda_T)} \Big|_{M(\lambda_T)^A_{\lambda_T-\nu}} \right]$

5.7: Proof of Jantzen's theorem from the key lemma

Definitions:

- $D_\nu(\lambda_T) := \det \left[(\cdot,\cdot)_{M(\lambda_T)} \Big|_{M(\lambda_T)^A_{\lambda_T - \nu}} \right]$
- Kostant partition function:
  $\mathcal{P} : \Lambda \to \mathbb{Z}^{\geq 0} : \lambda \mapsto \#(c_\alpha)_{\alpha \in \Phi^+}$ in $\mathbb{Z}^{\geq 0}$ such that $\lambda = -\sum_{\alpha \in \Phi^+} c_\alpha \alpha$

5.7: Proof of Jantzen's theorem from the key lemma

Definitions:

- $D_\nu(\lambda_T) := \det \left[(\cdot,\cdot)_{M(\lambda_T)} \Big|_{M(\lambda_T)^A_{\lambda_T - \nu}} \right]$
- Kostant partition function:
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Proposition 5.7:

$$D_\nu(\lambda_T) = \prod_{\alpha \in \Phi^+} \prod_{r=1}^{\infty} \left(\langle \lambda_T + \rho, \alpha^\vee \rangle - r \right)^{\mathcal{P}(\nu - r\alpha)}$$

Proof: will follow from Theorem 5.8.

5.7: Proof of Jantzen's theorem from the key lemma

Corollary: For any $\nu \in \Gamma$:

$$\sum_{i=1}^{\infty} \dim(\overline{M(\lambda_T)^A_{\lambda_T-\nu}(i)}) = \sum_{\alpha \in \Phi^+_\lambda} \mathcal{P}(\nu - \langle \lambda + \rho, \alpha^\vee \rangle \alpha)$$

Proof:

5.7: Proof of Jantzen's theorem from the key lemma

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Proof:
We will compute $v_T(D_\nu(\lambda_T))$ in 2 different ways.

1. By Proposition 5.7: $v_T(D_\nu(\lambda_T)) = \sum_{\alpha \in \Phi^+} \sum_{r=1}^{\infty} \mathcal{P}(\nu - r\alpha)\, v_T(\langle \lambda_T + \rho, \alpha^\vee \rangle - r)$

5.7: Proof of Jantzen's theorem from the key lemma

Corollary: For any $\nu \in \Gamma$:

$$\sum_{i=1}^{\infty} \dim\left(\overline{M(\lambda_T)^A_{\lambda_T-\nu}(i)}\right) = \sum_{\alpha \in \Phi^+_\lambda} \mathcal{P}(\nu - \langle \lambda + \rho, \alpha^\vee \rangle \alpha)$$

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$$v_T\left(\langle \lambda_T + \rho, \alpha^\vee \rangle - r\right) = v_T\left(\underline{\langle \lambda + \rho, \alpha^\vee \rangle - r + T\langle \rho, \alpha^\vee \rangle}\right)$$
$$= \begin{cases} 0 & \text{if } \alpha \notin \Phi^+_\lambda \\ \delta_{r, \langle \lambda+\rho, \alpha^\vee \rangle} & \text{if } \alpha \in \Phi^+_\lambda \end{cases}$$

$\langle \lambda + \rho, \alpha^\vee \rangle = n \in \mathbb{Z}^{>0} \Rightarrow s_\alpha . \lambda < \lambda \Rightarrow \alpha \in \Phi^+_\lambda$

5.7: Proof of Jantzen's theorem from the key lemma

Corollary: For any $\nu \in \Gamma$:

$$\sum_{i=1}^{\infty} \dim(\overline{M(\lambda_T)^A_{\lambda_T-\nu}(i)}) = \sum_{\alpha \in \Phi^+_\lambda} \mathcal{P}(\nu - \langle \lambda + \rho, \alpha^\vee \rangle \alpha)$$

Proof:

We will compute $v_T(D_\nu(\lambda_T))$ in 2 different ways.

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$$v_T\left(\langle \lambda_T + \rho, \alpha^\vee \rangle - r\right) = v_T\left(\langle \lambda + \rho, \alpha^\vee \rangle - r + T\langle \rho, \alpha^\vee \rangle\right)$$
$$= \begin{cases} 0 & \text{if } \alpha \notin \Phi^+_\lambda \\ \delta_{r,\langle \lambda+\rho, \alpha^\vee \rangle} & \text{if } \alpha \in \Phi^+_\lambda \end{cases}$$
$$\Rightarrow v_T(D_\nu(\lambda_T)) = \sum_{\alpha \in \Phi^+_\lambda} \mathcal{P}(\nu - \langle \lambda + \rho, \alpha^\vee \rangle \alpha)$$

5.7: Proof of Jantzen's theorem from the key lemma

Corollary: For any $\nu \in \Gamma$:

$$\sum_{i=1}^{\infty} \dim(\overline{M(\lambda_T)^A_{\lambda_T-\nu}(i)}) = \sum_{\alpha\in\Phi^+_\lambda} \mathcal{P}(\nu - \langle \lambda+\rho, \alpha^\vee\rangle\alpha)$$

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We will compute $v_T(D_\nu(\lambda_T))$ in 2 different ways.

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   $$v_T\left(\langle \lambda_T+\rho, \alpha^\vee\rangle - r\right) = v_T\left(\langle \lambda+\rho, \alpha^\vee\rangle - r + T\langle \rho, \alpha^\vee\rangle\right)$$
   $$= \begin{cases} 0 & \text{if } \alpha \notin \Phi^+_\lambda \\ \delta_{r,\langle\lambda+\rho,\alpha^\vee\rangle} & \text{if } \alpha \in \Phi^+_\lambda \end{cases}$$
   $$\Rightarrow v_T(D_\nu(\lambda_T)) = \sum_{\alpha\in\Phi^+_\lambda} \mathcal{P}(\nu - \langle \lambda+\rho, \alpha^\vee\rangle\alpha)$$

2. By Lemma 5.6.1: $v_T(D_\nu(\lambda_T)) = \sum_{i=1}^{\infty} \dim(\overline{M(\lambda_T)^A_{\lambda_T-\nu}(i)})$ $\square$

5.7: Proof of Jantzen's theorem from the key lemma

Theorem 5.3 [Jantzen]:

$M(\lambda)$ has a filtration
$M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \cdots \supset M(\lambda)^n \supset M(\lambda)^{n+1} = \{0\}$
such that

3) Jantzen sum formula: $\sum_{i=1}^{n} \mathrm{ch}\ M(\lambda)^i = \sum_{\alpha \in \Phi_\lambda^+} \mathrm{ch}\ M(s_\alpha \cdot \lambda)$

Proof:

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Proof: $\sum_{i=1}^{\infty} \mathrm{ch}\ M(\lambda)^i = \sum_{i=1}^{\infty} \sum_{\nu \in \Gamma} \dim(M(\lambda)^i_{\lambda-\nu})\, e(\lambda - \nu)$

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Proof: $\displaystyle\sum_{i=1}^{\infty} \mathrm{ch}\ M(\lambda)^i = \sum_{i=1}^{\infty} \sum_{\nu \in \Gamma} \dim(M(\lambda)^i_{\lambda-\nu})\, e(\lambda-\nu)$

$$= \sum_{i=1}^{\infty} \sum_{\nu \in \Gamma} \dim \left(\overline{M(\lambda_T)^A_{\lambda_T-\nu}(i)} \right) \Big|_{(T=0)} e(\lambda-\nu)$$

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Proof:
$$\begin{aligned}\sum_{i=1}^{\infty} \mathrm{ch}\ M(\lambda)^i &= \sum_{i=1}^{\infty} \sum_{\nu \in \Gamma} \dim(M(\lambda)^i_{\lambda-\nu})\, e(\lambda-\nu) \\ &= \sum_{i=1}^{\infty} \sum_{\nu \in \Gamma} \underline{\dim\left(\overline{M(\lambda_T)^A_{\lambda_T-\nu}(i)}\right)\Big|_{T=0}}\, e(\lambda-\nu) \\ &\overset{\text{Cor. 5.7}}{=} \sum_{\nu \in \Gamma} \sum_{\alpha \in \Phi_\lambda^+} \mathcal{P}(\nu - \langle \lambda+\rho, \alpha^\vee \rangle \alpha)\, e(\lambda-\nu)\end{aligned}$$

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Proof:
$$\sum_{i=1}^{\infty} \mathrm{ch}\ M(\lambda)^i = \sum_{i=1}^{\infty} \sum_{\nu \in \Gamma} \dim(M(\lambda)^i_{\lambda-\nu})\, e(\lambda - \nu)$$
$$= \sum_{i=1}^{\infty} \sum_{\nu \in \Gamma} \dim \left(\overline{M(\lambda_T)^A_{\lambda_T - \nu}(i)} \right) \Big|_{T=0} e(\lambda - \nu)$$
$$\overset{\text{Cor. 5.7}}{=} \sum_{\nu \in \Gamma} \sum_{\alpha \in \Phi_\lambda^+} \mathcal{P}(\nu - \langle \lambda + \rho, \alpha^\vee \rangle \alpha)\, e(\lambda - \nu)$$

With new summation index $\nu' = \nu - \langle \lambda + \rho, \alpha^\vee \rangle \alpha \in \Gamma$:
$$\sum_{i=1}^{\infty} \mathrm{ch}\ M(\lambda)^i = \sum_{\nu' \in \Gamma} \sum_{\alpha \in \Phi_\lambda^+} \mathcal{P}(\nu')\, e(\underbrace{\lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha}_{\|} - \nu')$$
$$= \sum_{\nu' \in \Gamma} \sum_{\alpha \in \Phi_\lambda^+} \mathcal{P}(\nu')\, e(\overbrace{s_\alpha \cdot \lambda} - \nu')$$

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Proof (cntd):
By Proposition 1.16: $\mathrm{ch}\ M(s_\alpha \cdot \lambda) = \mathcal{P} * e(s_\alpha \cdot \lambda) = \sum_{\nu \in \Gamma} \mathcal{P}(\nu) e(s_\alpha \cdot \lambda - \nu)$

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3) Jantzen sum formula: $\sum_{i=1}^{n} \mathrm{ch}\ M(\lambda)^i = \sum_{\alpha \in \Phi_\lambda^+} \mathrm{ch}\ M(s_\alpha \cdot \lambda)$

Proof (cntd):

By Proposition 1.16: $\mathrm{ch}\ M(s_\alpha \cdot \lambda) = \mathcal{P} * e(s_\alpha \cdot \lambda) = \sum_{\nu \in \Gamma} \mathcal{P}(\nu) e(s_\alpha \cdot \lambda - \nu)$

$$\Rightarrow \sum_{i=1}^{n} \mathrm{ch}\ M(\lambda)^i \overset{!}{=} \sum_{\alpha \in \Phi_\lambda^+} \mathrm{ch}\ M(s_\alpha \cdot \lambda) \qquad \square$$

5.7: Proof of Jantzen's theorem from the key lemma

Theorem 5.3 [Jantzen]:

$M(\lambda)$ has a filtration
$M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \cdots \supset M(\lambda)^n \supset M(\lambda)^{n+1} = \{0\}$
such that

3) Jantzen sum formula: $\sum_{i=1}^{n} \mathrm{ch}\ M(\lambda)^i = \sum_{\alpha \in \Phi_\lambda^+} \mathrm{ch}\ M(s_\alpha \cdot \lambda)$

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By Proposition 1.16: $\mathrm{ch}\ M(s_\alpha \cdot \lambda) = \mathcal{P} * e(s_\alpha \cdot \lambda) = \sum_{\nu \in \Gamma} \mathcal{P}(\nu) e(s_\alpha \cdot \lambda - \nu)$

$$\Rightarrow \sum_{i=1}^{n} \mathrm{ch}\ M(\lambda)^i \overset{!}{=} \sum_{\alpha \in \Phi_\lambda^+} \mathrm{ch}\ M(s_\alpha \cdot \lambda) \qquad \square$$

What if instead of $\lambda_T = \lambda + T\rho$ we choose $\lambda + T\alpha$ for $\alpha \neq \rho$? See Chapter 8.

$\hookrightarrow \langle \rho, \beta^\vee \rangle \neq 0, \ \forall \beta \in \Phi$

5.8: The Shapovalov determinant formula

Definitions: For any $\lambda \in \mathfrak{h}^*$

- Universal bilinear form
  $C : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{h}) : u \otimes u' \mapsto C(u, u') = \mathrm{proj}|_{U(\mathfrak{h})}(\mathcal{T}(u)u')$
  By Section 3.15: $(v, v')_{M(\lambda)} = \underset{\substack{\downarrow \\ U(\mathfrak{h}) \to \mathbb{C}}}{\lambda} \circ C(u, u')$ if $v = u \cdot v_\lambda^+, v' = u' \cdot v_\lambda^+$

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  By Section 3.15: $\underbrace{(v, v')_{M(\lambda)} = \lambda \circ C(u, u')}$ if $v = u \cdot v_\lambda^+, v' = u' \cdot v_\lambda^+$
- $\widetilde{(\cdot, \cdot)}_{M(\lambda)} : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to \mathbb{C} : u \otimes u' \mapsto (u \cdot v_\lambda^+, u' \cdot v_\lambda^+)_{M(\lambda)}$
  $\Rightarrow \widetilde{(\cdot, \cdot)}_{M(\lambda)} = \lambda \circ C$

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  $\Rightarrow \widetilde{(\cdot,\cdot)}_{M(\lambda)} = \lambda \circ C$
- For any $\alpha \in \Phi^+$: $h_\alpha \in \mathfrak{h}$ is such that $\lambda(h_\alpha) = \langle \lambda, \alpha^\vee \rangle$, $\forall \lambda \in \mathfrak{h}^*$

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- $U(\mathfrak{n}^-)_{-\nu} = \{y \in U(\mathfrak{n}^-) : [h, y] = -\nu(h)\, y,\ \forall h \in \mathfrak{h}\}$

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  $\Rightarrow$ if $\{g_i^\nu\}_i$ is basis for $U(\mathfrak{n}^-)_{-\nu}$, then $\{g_i^\nu \cdot v_\lambda^+\}_i$ is basis for $M(\lambda)_{\lambda-\nu}$

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- $C_\nu :=$ restriction of C to $U(\mathfrak{n}^-)_{-\nu}^{\otimes 2}$

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- Shapovalov matrix S_ν: matrix of C_ν w.r.t. chosen ordered basis for $U(\mathfrak{n}^-)_{-\nu}$

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For any $\nu \in \Gamma$:

- $U(\mathfrak{n}^-)_{-\nu} = \{y \in U(\mathfrak{n}^-) : [h,y] = -\nu(h)\, y, \ \forall h \in \mathfrak{h}\}$
  $\Rightarrow$ if $\{g_i^\nu\}_i$ is basis for $U(\mathfrak{n}^-)_{-\nu}$, then $\{g_i^\nu \cdot v_\lambda^+\}_i$ is basis for $M(\lambda)_{\lambda-\nu}$
- $C_\nu :=$ restriction of C to $U(\mathfrak{n}^-)_{-\nu}^{\otimes 2}$
- Shapovalov matrix S_ν: matrix of C_ν w.r.t. chosen ordered basis for $U(\mathfrak{n}^-)_{-\nu}$
- $D_\nu := \det(S_\nu) \in U(\mathfrak{h})$, well-defined up to non-zero scalar factor

5.8: The Shapovalov determinant formula

Theorem 5.8: For any $\nu \in \Gamma$:

$$D_\nu = \prod_{\alpha \in \Phi^+} \prod_{r=1}^{\infty} \left(h_\alpha + \langle \rho, \alpha^\vee \rangle - r\right)^{\mathcal{P}(\nu - r\alpha)}$$

Proof: will be given soon.

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Proposition 5.7:

$$D_\nu(\lambda_T) = \prod_{\alpha\in\Phi^+} \prod_{r=1}^{\infty} (\langle \lambda_T + \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)}$$

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Proof:

$$D_\nu(\lambda_T) = \det\left[(\cdot,\cdot)_{M(\lambda_T)}\Big|_{M(\lambda_T)^A_{\lambda_T - \nu}}\right] = \det\left[\widetilde{(\cdot,\cdot)}_{M(\lambda_T)}\Big|_{U(\mathfrak{n}_A^-)^{\otimes 2}_{-\nu}}\right]$$

$$M(\lambda_T)^A_{\lambda_T - \nu} = U(\mathfrak{n}_A^-)_{-\nu} \cdot v^+_{\lambda_T}$$

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Proof:

$$\begin{aligned} D_\nu(\lambda_T) &= \det\left[(\cdot,\cdot)_{M(\lambda_T)}\Big|_{M(\lambda_T)^A_{\lambda_T-\nu}}\right] = \det\left[\widetilde{(\cdot,\cdot)}_{M(\lambda_T)}\Big|_{U(\mathfrak{n}_A^-)^{\otimes 2}_{-\nu}}\right] \\ &= \det\left[\lambda_T \circ C\Big|_{U(\mathfrak{n}_A^-)^{\otimes 2}_{-\nu}}\right] = \lambda_T \circ \det\left[C\Big|_{U(\mathfrak{n}_A^-)^{\otimes 2}_{-\nu}}\right] \end{aligned}$$

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Proof: will be given soon.

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$$D_\nu(\lambda_T) = \prod_{\alpha \in \Phi^+} \prod_{r=1}^{\infty} \left(\langle \lambda_T + \rho, \alpha^\vee \rangle - r\right)^{\mathcal{P}(\nu - r\alpha)}$$

Proof:

$$\begin{aligned} D_\nu(\lambda_T) &= \det \left[(\cdot,\cdot)_{M(\lambda_T)} \Big|_{M(\lambda_T)^A_{\lambda_T - \nu}} \right] = \det \left[\widetilde{(\cdot,\cdot)}_{M(\lambda_T)} \Big|_{U(\mathfrak{n}_A^-)^{\otimes 2}_{-\nu}} \right] \\ &= \det \left[\lambda_T \circ C \Big|_{U(\mathfrak{n}_A^-)^{\otimes 2}_{-\nu}} \right] = \lambda_T \circ \det \left[C \Big|_{U(\mathfrak{n}_A^-)^{\otimes 2}_{-\nu}} \right] \\ &= \lambda_T \circ D_\nu = \prod_{\alpha \in \Phi^+} \prod_{r=1}^{\infty} \left(\lambda_T(h_\alpha) + \langle \rho, \alpha^\vee \rangle - r\right)^{\mathcal{P}(\nu - r\alpha)} \\ &= \prod_{\alpha \in \Phi^+} \prod_{r=1}^{\infty} \left(\langle \lambda_T + \rho, \alpha^\vee \rangle - r\right)^{\mathcal{P}(\nu - r\alpha)} \end{aligned}$$

$\square$

Exercise 5.8:

Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ with $\Delta = \{\alpha, \beta\}$ and $\nu = \gamma = \alpha + \beta$. Check that the matrix S_ν w.r.t. the ordered basis $\{y_\alpha y_\beta, y_\gamma\}$ of $U(\mathfrak{n}^-)_{-\nu}$ is given by

$$S_\nu = \begin{pmatrix} h_\alpha h_\beta + h_\beta & -h_\beta \\ -h_\beta & h_\alpha + h_\beta \end{pmatrix} \text{ with } D_\nu = h_\alpha h_\beta(h_\alpha + h_\beta + 1)$$

$$S_\nu = \begin{pmatrix} C(y_\alpha y_\beta, y_\alpha y_\beta) & C(y_\alpha y_\beta, y_\gamma) \\ C(y_\gamma, y_\alpha y_\beta) & C(y_\gamma, y_\gamma) \end{pmatrix}.$$

$$C(y_\alpha y_\beta, y_\alpha y_\beta) = \mathrm{proj}|_{U(\mathfrak{h})}(x_\beta \underbrace{x_\alpha y_\alpha}_{= \cancel{y_\alpha x_\alpha} + h_\alpha} y_\beta)$$

$$= \mathrm{proj}(x_\beta \underbrace{h_\alpha y_\beta}_{= y_\beta h_\alpha - \underbrace{\langle \beta, \alpha^\vee \rangle}_{= -1} y_\beta})$$

$$= \mathrm{proj}(\underbrace{x_\beta y_\beta}_{= \cancel{y_\beta x_\beta} + h_\beta}(h_\alpha + 1)) = h_\beta(h_\alpha + 1)$$

$\rho = \alpha + \beta$

$\langle \rho, \alpha^\vee \rangle = 2 + (-1) = 1,$

$\langle \rho, \beta^\vee \rangle = 1, \quad \langle \rho, (\alpha + \beta)^\vee \rangle = 2$

$\mathcal{P}(\alpha + \beta - r\alpha) = \delta_{r,1} = \mathcal{P}(\alpha + \beta - r\beta) = \mathcal{P}(\alpha + \beta - r(\alpha + \beta))$

$$\overset{\mathrm{Th.\,5.8}}{\Longrightarrow} D_\nu = (h_\alpha + 1 - 1)^1 \cdot (h_\beta + 1 - 1)^1 \cdot (h_\alpha + h_\beta + 2 - 1)^1 \overset{!}{=} h_\alpha h_\beta (h_\alpha + h_\beta + 1).$$

5.9: Shapovalov's proof of the determinant formula

Let $\Phi^+ = \{\alpha_1, \ldots, \alpha_m\}$, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$
PBW-basis for $U(\mathfrak{g})$: $y_1^{r_1} \ldots y_m^{r_m} h_1^{s_1} \ldots h_\ell^{s_\ell} x_1^{t_1} \ldots x_m^{t_m}$
for $(r_1, \ldots, r_m, s_1, \ldots, s_\ell, t_1, \ldots, t_m) \in$ certain subset of $(\mathbb{Z}^{\geq 0})^{\times(2m+\ell)}$

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Notations: For any $\nu \in \Gamma$

- A~~ny~~ basis of $U(\mathfrak{n}^-)_{-\nu}$ is of the form
  $\{y_\omega = y_1^{r_1} \ldots y_m^{r_m} : \omega = (r_1, \ldots, r_m) \in \Omega_\nu\}$,
  where $\Omega_\nu = \{(r_1, \ldots, r_m) \in (\mathbb{Z}^{\geq 0})^{\times m} : \sum_{i=1}^m r_i \alpha_i = \nu\}$

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- $d(\omega) = \sum_{i=1}^m r_i$

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- $d(\omega) = \sum_{i=1}^m r_i$
- $S_\nu = (c(\omega, \omega'))_{\omega, \omega' \in \Omega_\nu}$,
  where $c(\omega, \omega') = C_\nu(y_\omega, y_{\omega'}) = \mathrm{proj}|_{U(\mathfrak{h})}(\tau(y_\omega)y_{\omega'})$
  $= \mathrm{proj}|_{U(\mathfrak{h})}(x_m^{r_m} \ldots x_1^{r_1} y_1^{s_1} \ldots y_m^{s_m})$ if $\omega = (r_1, \ldots, r_m)$, $\omega' = (s_1, \ldots, s_m)$

5.9: Shapovalov's proof of the determinant formula

Lemma 5.9.1: For any $\nu \in \Gamma$:

$\forall k \in \{1, \dots, m\} : \forall \omega \in \Omega_\nu : \exists n \in \mathbb{Z}^{\geq 0} : \exists \omega_1, \dots, \omega_n \in \bigoplus_{\nu' \in \Gamma} \Omega_{\nu'}$:
$\exists p_1(\omega), \dots, p_n(\omega)$ polynomials of degree ≤ 1 in the h_α, $\alpha \in \Phi^+$:

$$x_k y_\omega = \sum_{i=1}^n p_i(\omega)\, y_{\omega_i} \mod U(\mathfrak{g})\mathfrak{n}^+$$

$\to p(\omega) = a_1 h_{\alpha_1} + a_2 h_{\alpha_2} + \dots + b$

Proof:

5.9: Shapovalov's proof of the determinant formula

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Proof: By induction on $d(\omega)$:

- If $d(\omega) = 0$, then $y_\omega = 1 \Rightarrow x_k y_\omega \in U(\mathfrak{g})\mathfrak{n}^+$

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$\forall k \in \{1, \dots, m\} : \forall \omega \in \Omega_\nu : \exists n \in \mathbb{Z}^{\geq 0} : \exists \omega_1, \dots, \omega_n \in \bigoplus_{\nu' \in \Gamma} \Omega_{\nu'}$:
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$$x_k y_\omega = \sum_{i=1}^n p_i(\omega)\, y_{\omega_i} \mod U(\mathfrak{g})\mathfrak{n}^+$$

Proof: By induction on $d(\omega)$:

- If $d(\omega) = 0$, then $y_\omega = 1 \Rightarrow x_k y_\omega \in U(\mathfrak{g})\mathfrak{n}^+$
- Let $d(\omega) > 0$. Write $\omega = (r_1, \dots, r_m)$. Let j be minimal such that $r_j \neq 0$. Set $\omega^\circ = (r_1, \dots, r_{j-1}, r_j - 1, r_{j+1}, \dots, r_m)$. Note: $d(\omega^\circ) < d(\omega)$

Lemma 5.9.1: For any $\nu \in \Gamma$:

$\forall k \in \{1, \ldots, m\} : \forall \omega \in \Omega_\nu : \exists n \in \mathbb{Z}^{\geq 0} : \exists \omega_1, \ldots, \omega_n \in \bigoplus_{\nu' \in \Gamma} \Omega_{\nu'}$:
$\exists p_1(\omega), \ldots, p_n(\omega)$ polynomials of degree ≤ 1 in the h_α, $\alpha \in \Phi^+$:

$$x_k y_\omega = \sum_{i=1}^n p_i(\omega)\, y_{\omega_i} \mod U(\mathfrak{g})\mathfrak{n}^+$$

Proof: By induction on $d(\omega)$:

- If $d(\omega) = 0$, then $y_\omega = 1 \Rightarrow x_k y_\omega \in U(\mathfrak{g})\mathfrak{n}^+$
- Let $d(\omega) > 0$. Write $\omega = (r_1, \ldots, r_m)$. Let j be minimal such that $r_j \neq 0$. Set $\omega^\circ = (r_1, \ldots, r_{j-1}, r_j - 1, r_{j+1}, \ldots, r_m)$. Note: $d(\omega^\circ) < d(\omega)$
$x_k y_\omega = x_k y_j y_{\omega^\circ} = y_j x_k y_{\omega^\circ} + [x_k, y_j] y_{\omega^\circ}$.

5.9: Shapovalov's proof of the determinant formula

Lemma 5.9.1: For any $\nu \in \Gamma$:

$\forall k \in \{1, \ldots, m\} : \forall \omega \in \Omega_\nu : \exists n \in \mathbb{Z}^{\geq 0} : \exists \omega_1, \ldots, \omega_n \in \bigoplus_{\nu' \in \Gamma} \Omega_{\nu'}$:
$\exists p_1(\omega), \ldots, p_n(\omega)$ polynomials of degree ≤ 1 in the h_α, $\alpha \in \Phi^+$:

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$x_k y_\omega = x_k y_j y_{\omega^\circ} = y_j x_k y_{\omega^\circ} + [x_k, y_j] y_{\omega^\circ}$.
    - By the IH: $\exists \omega_i, p_i(\omega^\circ)$: $x_k y_{\omega^\circ} = \sum_{i=1}^n p_i(\omega^\circ) y_{\omega_i}$ mod $U(\mathfrak{g})\mathfrak{n}^+$.

5.9: Shapovalov's proof of the determinant formula

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    If $p_i(\omega^\circ) = a_i h_{\alpha^{(i)}} + b_i \Rightarrow y_j p_i(\omega^\circ) = p_i'(\omega^\circ) y_j$,
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    where $p_i'(\omega^\circ) = a_i h_{\alpha^{(i)}} + b_i + \langle \alpha_j, \alpha^{(i)\vee} \rangle$
    $\Rightarrow y_j x_k y_{\omega^\circ} = \sum_{i=1}^n p_i'(\omega^\circ) y_j y_{\omega_i} \mod U(\mathfrak{g})\mathfrak{n}^+$

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    where $p_i'(\omega^\circ) = a_i h_{\alpha^{(i)}} + b_i + \langle \alpha_j, \alpha^{(i)\vee} \rangle$
    $\Rightarrow y_j x_k y_{\omega^\circ} = \sum_{i=1}^n p_i'(\omega^\circ) y_j y_{\omega_i} \mod U(\mathfrak{g})\mathfrak{n}^+$
    Also $y_j y_{\omega_i} \in U(\mathfrak{n}^-)$, so can be written as $\mathbb{C}$-linear combination of $y_{\omega'}$

5.9: Shapovalov's proof of the determinant formula

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$$x_k y_\omega = \sum_{i=1}^n p_i(\omega)\, y_{\omega_i} \quad \text{mod } U(\mathfrak{g})\mathfrak{n}^+$$

Proof (cntd): By induction on $d(\omega)$:

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    Note: $[x_k, y_j] = \prod_{i=1}^{\ell} \left[x_i^{a_i^{(k)}}, y_i^{a_i^{(j)}}\right]$
    - If $\alpha_k - \alpha_j \in \Phi^-$, then $\forall i : a_i^{(k)} \leq a_i^{(j)}$ with $<$ at least once
      $\Rightarrow \left[x_i^{a_i^{(k)}}, y_i^{a_i^{(j)}}\right] \in h_i^{a_i^{(k)}} y_i^{a_i^{(j)} - a_i^{(k)}} \mod U(\mathfrak{g})\mathfrak{n}^+$
      $\Rightarrow [x_k, y_j] = h_{\alpha_k} y_{\alpha_j - \alpha_k}$

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      $\Rightarrow [x_k, y_j] y_{\omega^\circ} = h_{\alpha_k} y_{\alpha_j - \alpha_k} y_{\omega^\circ}$ is $\mathbb{C}$-linear combination of $h_{\alpha_k} y_{\omega'}$

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      $\Rightarrow [x_k, y_j] y_{\omega^\circ} = h_{\alpha_j} x_{\alpha_k - \alpha_j} y_{\omega^\circ} \overset{\mathrm{IH}}{=} h_{\alpha_j} \sum_{i=1}^{n'} p_i''(\omega^\circ) y_{\omega_i'}$

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      $\Rightarrow \left[x_i^{a_i^{(k)}}, y_i^{a_i^{(j)}}\right] \in h_i^{a_i^{(j)}} x_i^{a_i^{(k)} - a_i^{(j)}} \mod U(\mathfrak{g})\mathfrak{n}^+$
      $\Rightarrow [x_k, y_j] y_{\omega^\circ} = h_{\alpha_j} x_{\alpha_k - \alpha_j} y_{\omega^\circ} \overset{\text{IH}}{=} h_{\alpha_j} \sum_{i=1}^{n'} p_i''(\omega^\circ) y_{\omega_i'}$
      Note: If $p_i''(\omega^\circ) = a_i h_{\alpha^{(i)}} + b_i \Rightarrow h_{\alpha_j} p_i''(\omega^\circ) = a_i h_{\alpha^{(i)} + \alpha_j} + b_i h_{\alpha_j}$ $\square$

5.9: Shapovalov's proof of the determinant formula

Lemma 5.9.2: For any $\nu \in \Gamma$, for any $\omega, \omega' \in \Omega_\nu$

The degree of $c(\omega, \omega')$, considered as a polynomial in the h_α, $\alpha \in \Phi^+$, satisfies

$$\deg(c(\omega, \omega')) \leq \min(d(\omega), d(\omega'))$$

Proof:

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Proof:

The bilinear form C is symmetric $\Rightarrow S_\nu$ is symmetric matrix

$\Rightarrow$ It suffices to show that $\deg(c(\omega, \omega')) \leq d(\omega)$.

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Proof:

The bilinear form C is symmetric $\Rightarrow S_\nu$ is symmetric matrix
$\Rightarrow$ It suffices to show that $\deg(c(\omega, \omega')) \leq d(\omega)$. By induction on $d(\omega)$.

- If $d(\omega) = 0 \Rightarrow y_\omega = 1 \Rightarrow c(\omega, \omega') = \mathrm{proj}|_{U(\mathfrak{h})}(y_{\omega'}) = \delta_{\omega',0} \Rightarrow \deg(c(\omega, \omega')) = 0$

5.9: Shapovalov's proof of the determinant formula

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The degree of $c(\omega, \omega')$, considered as a polynomial in the h_α, $\alpha \in \Phi^+$, satisfies

$$\deg(c(\omega, \omega')) \leq \min(d(\omega), d(\omega'))$$

Proof:

The bilinear form C is symmetric $\Rightarrow S_\nu$ is symmetric matrix

$\Rightarrow$ It suffices to show that $\deg(c(\omega, \omega')) \leq d(\omega)$. By induction on $d(\omega)$.

- If $d(\omega) = 0 \Rightarrow y_\omega = 1 \Rightarrow c(\omega, \omega') = \mathrm{proj}|_{U(\mathfrak{h})}(y_{\omega'}) = \delta_{\omega',0} \Rightarrow \deg(c(\omega, \omega')) = 0$
- If $d(\omega) > 0$: $y_\omega = y_j y_{\omega^\circ} \Rightarrow c(\omega, \omega') = \mathrm{proj}|_{U(\mathfrak{h})}(\tau(y_{\omega^\circ}) x_j y_{\omega'})$.

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  By Lemma 5.9.1: $\exists p_i, \omega_i$: $x_j y_{\omega'} = \sum_{i=1}^n y_{\omega_i} p_i(\omega')$ mod $U\xcancel{(\mathfrak{g})\mathfrak{n}^+}$

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  $\Rightarrow \deg(c(\omega, \omega')) \leq \underset{\max_i}{\cancel{\sum_{i=1}^n}} \deg(c(\omega^\circ, \omega_i)) + 1 \overset{\text{IH}}{\leq} d(\omega^\circ) + 1 = d(\omega)$ $\square$

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In fact Shapovalov showed: $\omega \neq \omega' \Rightarrow \deg(c(\omega, \omega')) < \min(d(\omega), d(\omega'))$

5.9: Shapovalov's proof of the determinant formula

Corollary 5.9.3: For any $\nu \in \Gamma$:

$$\deg(D_\nu) \leq \sum_{\alpha \in \Phi^+} \sum_{r=1}^{\infty} \mathcal{P}(\nu - r\alpha)$$

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Proof:

- $\deg(D_\nu) \leq \sum_{\omega \in \Omega_\nu} \max_{\omega' \in \Omega_\nu} (\deg(c(\omega, \omega'))) \overset{\text{Lemma 5.9.2}}{\leq} \sum_{\omega \in \Omega_\nu} d(\omega)$

$D_\nu = \det(c(\omega, \omega'))$

$\Rightarrow \deg(D_\nu) \leq \max_{\omega' \in \Omega_\nu} \deg\left(\prod_{\omega \in \Omega_\nu} c(\omega, \omega')\right)$

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- $$\sum_{\omega \in \Omega_\nu} d(\omega) = \sum_{(r_1,\ldots,r_m) \in \Omega_\nu} \sum_{i=1}^{m} r_i$$

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- $\sum_{\omega \in \Omega_\nu} d(\omega) = \sum_{(r_1, \dots, r_m) \in \Omega_\nu} \sum_{i=1}^{m} r_i$
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5.9: Shapovalov's proof of the determinant formula

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Proof (cntd):

- $\mathcal{P}(\nu - (r+1)\alpha_i) = \#\left[r_1, \dots, r_m \in \mathbb{Z}^{\geq 0} : \sum_{j=1}^{m} r_j \alpha_j = \nu - (r+1)\alpha_i\right]$

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  $= \mathcal{P}(\nu - r\alpha_i) - \#\left[r_1, \ldots, r_m \in \mathbb{Z}^{\geq 0} : \sum_{j=1}^{m} r_j\alpha_j = \nu - r\alpha_i, r_i = 0\right]$

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- $$\begin{aligned}\mathcal{P}(\nu-(r+1)\alpha_i) &= \#\Big[r_1,\ldots,r_m \in \mathbb{Z}^{\geq 0} : \sum_{j=1}^{m} r_j\alpha_j = \nu-(r+1)\alpha_i\Big] \\ &= \#\big[r_1,\ldots,r_m \in \mathbb{Z}^{\geq 0} : r_1\alpha_1+\cdots+(r_i-1)\alpha_i+\cdots+r_m\alpha_m = \nu-(r+1)\alpha_i, r_i \neq 0\big] \\ &= \#\big[r_1,\ldots,r_m \in \mathbb{Z}^{\geq 0} : r_1\alpha_1+\cdots+r_i\alpha_i+\cdots+r_m\alpha_m = \nu-r\alpha_i, r_i \neq 0\big] \\ &= \mathcal{P}(\nu-r\alpha_i) - \#\big[r_1,\ldots,r_m \in \mathbb{Z}^{\geq 0} : \textstyle\sum_{j=1}^{m} r_j\alpha_j = \nu-r\alpha_i, r_i=0\big] \\ &= \mathcal{P}(\nu-r\alpha_i) - \#\Big[r_1,\ldots,r_{i-1},r_{i+1},\ldots,r_m \in \mathbb{Z}^{\geq 0} : \sum_{\substack{j=1 \\ r_i=r}}^{m} r_j\alpha_j = \nu\Big]\end{aligned}$$

- $$\begin{aligned}\sum_{\omega\in\Omega_\nu} d(\omega) &= \sum_{i=1}^{m}\sum_{r=1}^{\infty} r\left[\mathcal{P}(\nu-r\alpha_i) - \mathcal{P}(\nu-(r+1)\alpha_i)\right] = \sum_{i=1}^{m}\sum_{r=1}^{\infty}\mathcal{P}(\nu-r\alpha_i) \\ &= \sum_{\alpha\in\Phi^+}\sum_{r=1}^{\infty}\mathcal{P}(\nu-r\alpha)\end{aligned}$$

$\square$

5.9: Shapovalov's proof of the determinant formula

Theorem 4.12: For any $\alpha \in \Phi^+$, $r \in \mathbb{Z}^{>0}$: $\exists$ Shapovalov elements $\theta_{\alpha,r} \in U(\mathfrak{b}^-)_{-r\alpha}$:

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- If $\alpha = \sum_{i=1}^{\ell} a_i \alpha_i$, then $\theta_{\alpha,r} = \overline{\theta_{\alpha,r}} + \sum_j p_j q_j$, where $\overline{\theta_{\alpha,r}} = \prod_{i=1}^{\ell} y_i^{r a_i}$, $p_j \in U(\mathfrak{n}^-)_{-r\alpha}$, $q_j \in U(\mathfrak{h})$ with $\deg(q_j) < r \sum_{i=1}^{\ell} a_i$

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Notations: For any $\nu \in \Gamma$, $\alpha \in \Phi^+$:

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  Since right multiplication in $U(\mathfrak{g})$ is injective:
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- Let $T \subset U(\mathfrak{n}^-)_{-\nu}$ be such that $\underline{U(\mathfrak{n}^-)_{-\nu}} = T \oplus V_{\alpha,r}$
  $\underline{V} := T \oplus V'_{\alpha,r} \subset U(\mathfrak{b}^-)_{-\nu}$

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  $V := T \oplus V'_{\alpha,r} \subset U(\mathfrak{b}^-)_{-\nu}$
- $S_V :=$ matrix of $C|_{V\otimes V}$

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  $V'_{\alpha,r} := U(\mathfrak{n}^-)_{-\nu+r\alpha}\, \theta_{\alpha,r} \subset U(\mathfrak{b}^-)_{-\nu}$
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  $\dim(V'_{\alpha,r}) = \dim(V_{\alpha,r}) = \dim(U(\mathfrak{n}^-)_{-\nu+r\alpha}) = \mathcal{P}(\nu - r\alpha)$
- Let $T \subset U(\mathfrak{n}^-)_{-\nu}$ be such that $U(\mathfrak{n}^-)_{-\nu} = T \oplus V_{\alpha,r}$
  $V := T \oplus V'_{\alpha,r} \subset U(\mathfrak{b}^-)_{-\nu}$
- $S_V :=$ matrix of $C|_{V \otimes V}$
- $D_V := \det(S_V)$

5.9: Shapovalov's proof of the determinant formula

$(U-I)^n = 0 \rightsquigarrow U = \begin{pmatrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \rightsquigarrow \det = 1$

Lemma 5.9.4:

$\exists$ unipotent matrix U: $S_V = U^T S_{\cancel{V}\nu} U$ and hence $D_\nu = D_V$ (up to scalar $\neq 0$)

Proof:

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$\exists$ unipotent matrix U: $S_V = U^T S_\nu U$ and hence $D_\nu = D_V$ (up to scalar $\neq 0$)

Proof:

- $$U(\mathfrak{n}^-)_{-\nu} \otimes U(\mathfrak{h}) \cong (T \oplus V_{\alpha,r}) \otimes U(\mathfrak{h}) \cong (T \otimes U(\mathfrak{h})) \oplus (V_{\alpha,r} \otimes U(\mathfrak{h}))$$
  $$\cong (T \otimes U(\mathfrak{h})) \oplus (V'_{\alpha,r} \otimes U(\mathfrak{h})) \cong (T \oplus V'_{\alpha,r}) \otimes U(\mathfrak{h}) \cong V \otimes U(\mathfrak{h})$$
  All isomorphisms are natural, i.e. either by multiplication or $\overline{\theta_{\alpha,r}} \mapsto \theta_{\alpha,r}$
- $\dim(V) = \dim(T) + \dim(V'_{\alpha,r}) = \dim(T) + \dim(V_{\alpha,r}) = \dim(U(\mathfrak{n}^-)_{-\nu})$
- $S_V = C|_{V^{\otimes 2}}$, $S_\nu = C|_{U(\mathfrak{n}^-)^{\otimes 2}_{-\nu}}$ $\square$

5.9: Shapovalov's proof of the determinant formula

Lemma 5.9.5: For any $\alpha \in \Phi^+$, $r \in \mathbb{Z}^{>0}$, $\omega \in \bigoplus_{\nu' \in \Gamma} \Omega_{\nu'}$, $y \in U(\mathfrak{n}^-)$:

$C(y_\omega, y\theta_{\alpha,r}) \in (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)\, U(\mathfrak{h})$

Proof:

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It suffices to show the claim for $y = y_{\omega'}$.

Let $y_\omega = y_j y_{\omega^\circ}$, $y = y_{\omega'} = y_k y_{\omega'^\circ}$.

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Let $y_\omega = y_j y_{\omega^\circ}$, $y = y_{\omega'} = y_k y_{\omega'^\circ}$.

$C(y_\omega, y\theta_{\alpha,r}) = \mathrm{proj}|_{U(\mathfrak{h})}(\tau(y_\omega) y\theta_{\alpha,r}) = \mathrm{proj}|_{U(\mathfrak{h})}(\tau(y_{\omega^\circ})\underbrace{x_j y_{\omega'}\theta_{\alpha,r}})$.

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We prove that $x_j y_{\omega'}\theta_{\alpha,r} \in \underbrace{U(\mathfrak{g})(h_\alpha + \langle \rho, \alpha^\vee \rangle - r) + U(\cancel{\mathfrak{g}})\mathfrak{n}^+ + U(\cancel{\mathfrak{g}})\mathfrak{n}^-}$,
by induction on $d(\omega')$:

5.9: Shapovalov's proof of the determinant formula

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- If $d(\omega') = 0$, then the claim follows from Theorem 4.12.

5.9: Shapovalov's proof of the determinant formula

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- If $d(\omega') = 0$, then the claim follows from Theorem 4.12.
- If $d(\omega') > 0$, then $x_j y_{\omega'} \theta_{\alpha,r} = x_j y_k y_{\omega'^\circ} \theta_{\alpha,r} = y_k x_j y_{\omega'^\circ} \theta_{\alpha,r} + [x_j, y_k] y_{\omega'^\circ} \theta_{\alpha,r}$

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$C(y_\omega, y\theta_{\alpha,r}) = \mathrm{proj}|_{U(\mathfrak{h})}\left(\tau(y_\omega) y \theta_{\alpha,r}\right) = \mathrm{proj}|_{U(\mathfrak{h})}\left(\tau(y_{\omega^\circ}) x_j y_{\omega'} \theta_{\alpha,r}\right)$.

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    - $y_k x_j y_{\omega'^\circ} \theta_{\alpha,r} \in U(\mathfrak{g})(h_\alpha + \langle \rho, \alpha^\vee \rangle - r) + U(\mathfrak{g})\mathfrak{n}^+ + U(\mathfrak{g})\mathfrak{n}^-$ by the IH

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Proof:

It suffices to show the claim for $y = y_{\omega'}$.
Let $y_\omega = y_j y_{\omega^\circ}$, $y = y_{\omega'} = y_k y_{\omega'^\circ}$.
$C(y_\omega, y\theta_{\alpha,r}) = \mathrm{proj}|_{U(\mathfrak{h})}\left(\tau(y_\omega) y \theta_{\alpha,r}\right) = \mathrm{proj}|_{U(\mathfrak{h})}\left(\tau(y_{\omega^\circ}) x_j y_{\omega'} \theta_{\alpha,r}\right)$.
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  - $y_k x_j y_{\omega'^\circ} \theta_{\alpha,r} \in U(\mathfrak{g})(h_\alpha + \langle \rho, \alpha^\vee \rangle - r) + U(\mathfrak{g})\mathfrak{n}^+ + U(\mathfrak{g})\mathfrak{n}^-$ by the IH
  - $[x_j, y_k] y_{\omega'^\circ} \theta_{\alpha,r} = \begin{cases} h_{\alpha_j} y_{\omega'^\circ} \theta_{\alpha,r} & \text{if } j = k \\ h_{\alpha_j} y_{\alpha_k - \alpha_j} y_{\omega'^\circ} \theta_{\alpha,r} & \text{if } \alpha_j - \alpha_k \in \Phi^- \\ h_{\alpha_k} x_{\alpha_j - \alpha_k} y_{\omega'^\circ} \theta_{\alpha,r} & \text{if } \alpha_j - \alpha_k \in \Phi^+ \end{cases}$

IH

□

5.9: Shapovalov's proof of the determinant formula

Lemma 5.9.6: For any $\nu \in \Gamma$:

$$D_\nu \in \prod_{\alpha \in \Phi^+} \prod_{r=1}^{\infty} (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)} U(\mathfrak{h})$$

Proof:

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Proof: Let $\alpha \in \Phi^+$ and $r \in \mathbb{Z}^{>0}$ be arbitrary.
By Lemma 5.9.4: $D_\nu = D_V = \det(S_V) = \det(C|_{V\otimes V})$. $V = T \oplus V'_{\alpha,r}$.

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- Basis elements of T: y_ω for certain $\omega \in \Omega_\nu$
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$\Rightarrow$ matrix entry of S_V on column indexed by $V'_{\alpha,r}$ and row indexed by T:

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$C(y_\omega, y\theta_{\alpha,r}) \overset{\text{Lemma 5.9.5}}{\in} (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)\, U(\mathfrak{h})$.

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$\Rightarrow$ matrix entry of S_V on column indexed by $V'_{\alpha,r}$ and row indexed by T:
$C(y_\omega, y\theta_{\alpha,r}) \overset{\text{Lemma 5.9.5}}{\in} (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)\, U(\mathfrak{h})$.
There are $\dim(V'_{\alpha,r}) = \mathcal{P}(\nu - r\alpha)$ such columns
$\Rightarrow D_\nu \in (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)} U(\mathfrak{h})$.

5.9: Shapovalov's proof of the determinant formula

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There are $\dim(V'_{\alpha,r}) = \mathcal{P}(\nu - r\alpha)$ such columns
$\Rightarrow D_\nu \in (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)} U(\mathfrak{h})$.
The factors $(h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)}$ are relatively prime
$\Rightarrow D_\nu \in \prod_{\alpha \in \Phi^+} \prod_{r=1}^{\infty} (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)} U(\mathfrak{h})$. □

5.9: Shapovalov's proof of the determinant formula

Theorem 5.8: For any $\nu \in \Gamma$:

$$D_\nu = \prod_{\alpha \in \Phi^+} \prod_{r=1}^{\infty} (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)}$$

Proof:

5.9: Shapovalov's proof of the determinant formula

Theorem 5.8: For any $\nu \in \Gamma$:

$$D_\nu = \prod_{\alpha \in \Phi^+} \prod_{r=1}^{\infty} (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)}$$

Proof:

By Lemma 5.9.5:

$$\sum_{\alpha \in \Phi^+} \sum_{r=1}^{\infty} \mathcal{P}(\nu - r\alpha) = \deg \Big(\prod_{\alpha \in \Phi^+} \prod_{r=1}^{\infty} (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)} \Big) \leq \deg(D_\nu)$$

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But also by Corollary 5.9.3:

$$\deg(D_\nu) \leq \sum_{\alpha \in \Phi^+} \sum_{r=1}^{\infty} \mathcal{P}(\nu - r\alpha)$$

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By Lemma 5.9.5:

$$\sum_{\alpha\in\Phi^+} \sum_{r=1}^{\infty} \mathcal{P}(\nu - r\alpha) = \deg\Big(\prod_{\alpha\in\Phi^+} \prod_{r=1}^{\infty} (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)} \Big) \leq \deg(D_\nu)$$

But also by Corollary 5.9.3:

$$\deg(D_\nu) \leq \sum_{\alpha\in\Phi^+} \sum_{r=1}^{\infty} \mathcal{P}(\nu - r\alpha)$$

$$\Rightarrow D_\nu = \prod_{\alpha\in\Phi^+} \prod_{r=1}^{\infty} (h_\alpha + \langle \rho, \alpha^\vee \rangle - r)^{\mathcal{P}(\nu - r\alpha)} \quad \text{up to non-zero scalar}$$

□