Content

In this chapter we will answer the following questions:

1) What are the simple submodules of $M(\lambda)$?
2) When is $M(\lambda)$ simple?
3) When does an embedding $M(\mu) \to M(\lambda)$ exist?
4) Can we construct such an embedding explicitly?
5) What are the blocks of $\mathcal{O}$?

Today: (1), (2) for $\lambda \in \Lambda$ and (3) for $\mu, \lambda \in \Lambda$.

Simple Submodules of Verma Modules

Lemma (4.1)

Any two nonzero left ideals of a left noetherian ring without zero divisors must intersect nontrivially.

Proposition (4.1)

For any $\lambda \in \mathfrak{h}^*$, the module $M(\lambda)$ has a unique simple submodule, which is therefore its socle.

Proof:

- $M(\lambda)$ is artinian, so it has a simple submodule.
- Suppose L, L' are distinct simple submodule of $M(\lambda)$, then $L \cap L' = \{0\}$.
- As $U(\mathfrak{n}^-)$-modules, $M(\lambda) \cong U(\mathfrak{n}^-)$. So L and L' are left ideals of $U(\mathfrak{n}^-)$.
- $U(\mathfrak{n}^-)$ is a left noetherian ring without zero divisors, so $L \cap L' \neq \{0\}$. This is a contradiction. □

Simple Submodules of Verma Modules

Note:

- The simple submodule of $M(\lambda)$ is isomorphic to some $L(\mu)$ with $\mu \leq \lambda$. Moreover, $\mu = w \cdot \lambda$ for some $w \in W_{[\lambda]}$.

Exercise:

- Let M be a nonzero submodule of $M(\lambda)$. Then M has a nondegenerate contravariant form if and only if it is the unique simple submodule of $M(\lambda)$.

Homomorphisms Between Verma Modules

Theorem (4.2)

Let $\lambda, \mu \in \mathfrak{h}^*$.

a) *Any nonzero homomorphism* $\varphi : M(\mu) \to M(\lambda)$ *is injective.*
b) *In all cases,* $\dim \operatorname{Hom}(M(\mu), M(\lambda)) \leq 1$.
c) *The unique simple submodule* $L(\mu)$ *in* $M(\lambda)$ *is a Verma module.*

Proof:

a) As a $U(\mathfrak{n}^-)$-module homomorphism $\varphi : U(\mathfrak{n}^-) \to U(\mathfrak{n}^-)$, $u' \mapsto u'u$, for some fixed $u \neq 0$.
   Since $U(\mathfrak{n}^-)$ has no zero-divisors, $\operatorname{Ker} \varphi = 0$.
b) Consider nonzero $\varphi_1, \varphi_2 : M(\mu) \to M(\lambda)$ and unique simple submodule $L \subset M(\mu)$. Then $\varphi_1(L) = \varphi_2(L)$ is simple.
   Let $\varphi_3 = \varphi_1 - c\varphi_2$ s.t. $\varphi_3(L) = \{0\}$. Then $\varphi_3 = 0$ by (a).
c) By universal property of $M(\mu)$, there exists $\varphi : M(\mu) \to M(\lambda)$, with $\varphi(M(\mu)) = L(\mu)$. Now φ is injective by (a). □

Homomorphisms Between Verma Modules

Notes:

- Whenever $\mathrm{Hom}(M(\mu), M(\lambda)) \neq 0$ we can now unambiguously write $M(\mu) \subset M(\lambda)$.
- One major goal in this chapter is to study this embedding.
    - When does it exist?
    - How can we construct it?
- The other goal is to determine when $M(\lambda)$ is simple.

Special Case: λ is a Dominant Integral Weight

Proposition (4.3)

Suppose $\lambda + \rho \in \Lambda^+$. Then $M(w \cdot \lambda) \subset M(\lambda)$, for all $w \in W$; thus all $[M(\lambda) : L(w \cdot \lambda)] > 0$.
More precisely, if w has reduced expression $w = s_n \cdots s_1$, with s_i reflection relative to the simple root α_i, then there is a sequence

$$M(w \cdot \lambda) = M(\lambda_n) \subset M(\lambda_{n-1}) \subset \cdots \subset M(\lambda_0) = M(\lambda),$$

where $\lambda_0 := \lambda$ and $\lambda_k := s_k \cdot \lambda_{k-1}$, for $k \in \{1, \ldots, n\}$.
In particular, $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_0$, with $\langle \lambda_k + \rho, \alpha_{k+1}^\vee \rangle \in \mathbb{Z}^+$, for $k = 0, \ldots, n-1$.

Proof:
Use the reformulated Proposition 1.4 and Theorem 4.2(c):

- If $\langle \lambda_k + \rho, \alpha_{k+1}^\vee \rangle \in \mathbb{Z}^+$, then there exists an embedding $M(s_{k+1} \cdot \lambda_k) = M(\lambda_{k+1}) \to M(\lambda_k)$. $\square$

Special Case: λ is a Dominant Integral Weight

Exercises: Assume $\lambda + \rho \in \Lambda^+$.

- The unique simple submodule of $M(\lambda)$ is isomorphic to $M(w_\circ \cdot \lambda)$.
- If $\lambda \in \Lambda^+$, then all inclusions in the proposition are proper.

Notes:

- This proposition will generalize as follow:

  Let $\lambda \in \mathfrak{h}^*$. Given $\alpha > 0$, suppose $\mu := s_\alpha \cdot \lambda \leq \lambda$. Then there exists an embedding $M(\mu) \subset M(\lambda)$.
- The failure of Proposition 1.4 to carry over to nonsimple positive roots, means that a totally different strategy is needed to generalize this result.

Simplicity Criterion: Integral Case

Recall that $\lambda \in \mathfrak{h}^*$ is called antidominant if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ for all $\alpha \in \Phi^+$.

Theorem (4.4, Simplicity Criterion)

Let $\lambda \in \mathfrak{h}^*$. Then $M(\lambda) = L(\lambda)$ if and only if λ is antidominant.

Proof (Integral case, $\lambda \in \Lambda$):

- Suppose $M(\lambda)$ is simple and that λ is not antidominant.
  - Then, since $\lambda \in \Lambda$, we can find a simple root α such that $\langle \lambda + \rho, \alpha^\vee \rangle > 0$.
  - Using the Proposition 1.4 and Theorem 4.2(c), we get a proper embedding $M(s_\alpha \cdot \lambda) \to M(\lambda)$.
  - This contradicts the simplicity of $M(\lambda)$.
- Suppose λ is antidominant.
  - Then by (3.5), $\lambda \leq w \cdot \lambda$.
  - But $M(\lambda)$ only has composition factors $L(w \cdot \lambda)$ with $w \cdot \lambda \leq \lambda$.
  - Therefore $L(\lambda)$ is the only composition factor and it occurs only once, so $M(\lambda) = L(\lambda)$. $\square$

Simplicity Criterion: Integral Case

Exercise:

- If $\lambda \in \Lambda$ is antidominant, then the socle of $P(w \cdot \lambda)$ with $w \in W$ is a direct sum of copies of $L(\lambda)$.
- The general version of this result is proven later.

Notes:

- Only the second part of this proof can be generalized to $\lambda \in \mathfrak{h}^*$.
- The first part does not generalize since Proposition 1.4 no longer applies.
- We therefore need more information on embeddings $M(s_\alpha \cdot \lambda) \subset M(\lambda)$, where α is not simple.

Existence of Embeddings: Preliminaries

Proposition (4.5)

Let $\mu, \lambda \in \mathfrak{h}^*$ *and* $\alpha \in \Delta$, *with* $n := \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}$ *and*

$$M(s_\alpha \cdot \mu) \subset M(\mu) \subset M(\lambda).$$

Then there are two possibilities for the position of $M(s_\alpha \cdot \lambda)$*:*

a) *If* $n \leq 0$, *then* $M(\lambda) \subset M(s_\alpha \cdot \lambda)$.

b) *If* $n > 0$, *then* $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda) \subset M(\lambda)$.

Lemma (4.5)

Let $\mathfrak{a}$ *be a nilpotent Lie algebra, with* $x \in \mathfrak{a}$ *and* $u \in U(\mathfrak{a})$. *Given a positive integer* n, *there exists an integer* t *depending on* x *and* u *such that* $x^t u \in U(\mathfrak{a})x^n$.

We also note that for any $\alpha \in \Delta$ and $t > 0$,

$$[x, y_\alpha^t] = t y_\alpha^{t-1}(h_\alpha - t + 1).$$

Existence of Embeddings: Preliminaries

Proof (Proposition 4.5):

a) If $n \leq 0$, then $M(\lambda) \subset M(s_\alpha \cdot \lambda)$ by Proposition 1.4, since

$$\langle s_\alpha \cdot \lambda + \rho, \alpha^\vee \rangle = \langle s_\alpha \cdot (\lambda + \rho), \alpha^\vee \rangle = \langle \lambda + \rho, -\alpha^\vee \rangle = -n \geq 0.$$

b) If $n > 0$, then we want $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda) \subset M(\lambda)$.

- Proposition 1.4 immediately gives $M(s_\alpha \cdot \lambda) \subset M(\mu)$.
- Letting $s := \langle \mu + \rho, \alpha^\vee \rangle$ we get maximal vectors

$$v_\lambda^+ \in M(\lambda), \quad y_\alpha^n \cdot v_\lambda^+ \in M(s_\alpha \cdot \lambda), \quad v_\mu^+ \in M(\mu), \quad y_\alpha^s \cdot v_\mu^+ \in M(s_\alpha \cdot \mu).$$

- Since $M(\mu) \subset M(\lambda)$, there is $u \in U(\mathfrak{n}^-)$ with $v_\mu^+ = u \cdot v_\lambda^+$.
- Lemma 4.5 gives us $t \geq s$ such that $y_\alpha^t u \in U(\mathfrak{n}^-) y_\alpha^n$. So

$$y_\alpha^t \cdot v_\mu^+ = y_\alpha^t u \cdot v_\lambda^+ \in U(\mathfrak{n}^-) y_\alpha^n v_\lambda^+ \subset M(s_\alpha \cdot \lambda).$$

- If $t > s$, then we use $[x, y_\alpha^t] = t y_\alpha^{t-1}(h_\alpha - t + 1)$ to get

$$(s - t) t y_\alpha^{t-1} v_\mu^+ = x_\alpha y_\alpha^t \cdot v_\mu^+ \in M(s_\alpha \cdot \lambda).$$

- This proves $y_\alpha^s \cdot v_\mu^+ \in M(s_\alpha \cdot \lambda)$ and $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda)$. □

Existence of Embeddings: Integral Case

Theorem (4.6, Verma)

Let $\lambda \in \mathfrak{h}^*$. Given $\alpha > 0$, suppose $\mu := s_\alpha \cdot \lambda \leq \lambda$. Then there exists an embedding $M(\mu) \subset M(\lambda)$.

Proof (Integral case, $\lambda \in \Lambda$):

- Since λ is integral, so is μ. Therefore we can find $w \in W$ such that $\mu' := w^{-1} \cdot \mu \in \Lambda^+ - \rho$.
- Considering a reduced expression $w = s_n \cdots s_1$, we define weights $\mu_0 := \mu'$ and $\mu_k := s_k \cdot \mu_{k-1}$, for $k = 1, \ldots, n$.
- Proposition 4.3 tells us that $\mu_0 \geq \cdots \geq \mu_n$ and that

$$M(\mu_0) \supset M(\mu_1) \supset \cdots \supset M(\mu_n).$$

- Letting $\lambda' := w^{-1} \cdot \lambda$ we define a parallel list of weights. $\lambda_0 := \lambda'$ and $\lambda_k := s_k \cdot \lambda_{k-1}$, for $k = 1, \ldots, n$.

Existence of Embeddings: Integral Case

Proof (continued):

- A short calculation shows that if $w_k := s_{k+1} \cdots s_n$, then $\mu_k = s_{\beta_k} \cdot \lambda_k$, β_k is the root with $s_{\beta_k} = w_k^{-1} s_\alpha w_k$.
- It follows that $\mu_k - \lambda_k \in \mathbb{Z}\beta_k$.
- We may assume that $\mu < \lambda$. This implies $\mu_k \neq \lambda_k$ and in particular $\mu' > \lambda'$, since μ' is dominant.
- There must thus be a least index k such that $\mu_k > \lambda_k$ and $\mu_{k+1} < \lambda_{k+1}$. We fix this k.
- We will prove $M(\mu_{k+1}) \subset M(\lambda_{k+1})$, $M(\mu_{k+2}) \subset M(\lambda_{k+2})$,... Culminating in $M(\mu_n) \subset M(\lambda_n)$.
- By definition $\mu_{k+1} - \lambda_{k+1} = s_{k+1}(\mu_k - \lambda_k)$.
- By our choice of k, we get $\mu_{k+1} - \lambda_{k+1} \in \mathbb{Z}^- \beta_{k+1}$ and $s_{k+1}(\mu_k - \lambda_k) \in \mathbb{Z}^+ \beta_k$. So $\beta_k = \beta_{k+1} = \alpha_{k+1}$.
- Prop 1.4 yields $M(\mu_{k+1}) = M(s_{k+1} \cdot \lambda_{k+1}) \subset M(\lambda_{k+1})$.

Existence of Embeddings: Integral Case

Proof (continued):

- Combined with the sequence of embeddings we get

$$M(\mu_{k+2}) = M(s_{k+2} \cdot \mu_{k+1}) \subset M(\mu_{k+1}) \subset M(\lambda_{k+1}).$$

- Proposition 4.5 then implies that

$$M(\mu_{k+2}) \subset M(s_{k+2} \cdot \lambda_{k+1}) = M(\lambda_{k+2}).$$

- Iterating these last arguments we get

$$M(\mu) = M(\mu_n) \subset M(\lambda_n) = M(\lambda).$$

□

Extra: Solution to problem we discussed at the end

Remark

Let $\lambda \in \mathfrak{h}^*$, $\alpha > 0$ and $M(s_\alpha \cdot \lambda) \subset M(\lambda)$. Then $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^+$.

Proof: The embedding implies that $s_\alpha \cdot \lambda \leq \lambda$.
In general we have

$$s_\alpha \cdot \lambda = s_\alpha(\lambda + \rho) - \rho = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha$$

So $s_\alpha \cdot \lambda \leq \lambda$ if and only if

$$\lambda - s_\alpha \cdot \lambda = \langle \lambda + \rho, \alpha^\vee \rangle \alpha \in \Gamma$$

if and only if

$$\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^+.$$

□

Existence of Embeddings: General Case

Theorem (4.6, Verma)

Let $\lambda \in \mathfrak{h}^*$. Given $\alpha > 0$, suppose $\mu := s_\alpha \cdot \lambda \leq \lambda$. Then there exists an embedding $M(\mu) \subset M(\lambda)$.

Proof:

- Note that $\nu := \lambda - \mu \in \Gamma$ and let

$$X := \{\lambda \in \mathfrak{h}^* : M(\mu) = M(\lambda - \nu) \subset M(\lambda)\}$$
$$H := \{\lambda \in \mathfrak{h}^* : \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^+\}$$

- Recall that $s_\alpha \cdot \lambda \leq \lambda$ if and only if $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^+$. It is enough to prove that $X = H$.
- We know that $X \subset H$ and that $\Lambda \cap H \subset X$.
- By 1.9 we know that $\Lambda \cap H$ is is Zariski dense in H.
- Proving that $X \subset \mathfrak{h}^*$ is Zariski closed implies $X = H$.

Existence of Embeddings: General Case

Proof(continued, X is Zariski closed):

- $X = \{\lambda \in \mathfrak{h}^* : M(\lambda - \nu) \subset M(\lambda)\}$.
- We need to construct a polynomial on $\mathfrak{h}^*$ whose set of common zeros is X.
- Write $\lambda = \lambda_1\overline{\omega}_1 + \cdots + \lambda_\ell\overline{\omega}_\ell$ and consider $\lambda_1, \ldots, \lambda_\ell$ as polynomial variables.
- We construct a linear map $g^\lambda : U(\mathfrak{n}^-)_{-\nu} \to U(\mathfrak{n}^-)^\ell$, such that
    - its matrix is written in terms of the λ_i.
    - $\operatorname{rank} g^\lambda < \dim U(\mathfrak{n}^-)_{-\nu}$ if and only if $\lambda \in X$.
- The matrix of g^λ then has a certain minor
    - which depends polynomially on the λ_i's.
    - whose determinant is 0 if and only if $\lambda \in X$.
- The construction of such a g^λ thus proves X to be Zariski closed.

Existence of Embeddings: General Case

Proof(continued, construction of g^λ):

- Let (h_i, x_i, y_i) be standard bases for $\mathfrak{s}_i \cong \mathfrak{sl}(2,\mathbb{C})$ corresponding to simple roots, for $i = 1, \ldots, \ell$.
- For $u \in U(\mathfrak{n}^-)$ we can find $u_i, u_i' \in U(\mathfrak{n}^-)$ depending linearly on u such that $[x_i, u] = u_i + u_i' h_i$.
- We define for $i = 1, \ldots, \ell$, the linear maps

$$f_i^\lambda : U(\mathfrak{n}^-)_{-\nu} \to U(\mathfrak{n}^-), \quad u \mapsto u_i + \lambda(h_i) u_i' = u_i + \lambda_i u_i'.$$

$$g^\lambda : U(\mathfrak{n}^-)_{-\nu} \to U(\mathfrak{n}^-)^\lambda, \quad u \mapsto f_1^\lambda(u) \oplus \cdots \oplus f_\ell^\lambda(u).$$

- Let $v^+ \in M(\lambda)$ be a maximal vector of weight λ.
- A short calculation shows that $g^\lambda(u) = 0$ if and only if $u \cdot v^+$ is a maximal vector of weight $\lambda - \nu$.
- So $\operatorname{rank} g^\lambda < \dim U(\mathfrak{n}^-)_{-\nu}$ if and only if $M(\lambda - \nu) \subset M(\lambda)$.

□

Existence of Embeddings: General Case

Notes:

- *Generalization of Proposition 1.4:*
  Let $\lambda \in \mathfrak{h}^*$ and $\alpha > 0$. Then $s_\alpha \cdot \lambda \leq \lambda$ if and only if $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^+$ if and only if $M(s_\alpha \cdot \lambda) \subset M(\lambda)$.
- *Generalization of Proposition 4.3:*
  Let $\lambda \in \mathfrak{h}^*$ and $\alpha_1, \ldots, \alpha_n > 0$ with $(s_{\alpha_n} \cdots s_{\alpha_1}) \cdot \lambda \leq \cdots \leq s_{\alpha_1} \cdot \lambda \leq \lambda$, then

$$M(\lambda) \supset M(s_{\alpha_1} \cdot \lambda) \supset \cdots \supset M((s_{\alpha_n} \cdots s_{\alpha_1}) \cdot \lambda).$$

- We thus have a sufficient condition for $[M(\lambda) : L(s_\alpha \cdot \lambda)] > 0$. This is also a necessary condition (5.1).

Simplicity Criterion: General Case

$\lambda \in \mathfrak{h}^*$ is antidominant if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ for all $\alpha \in \Phi^+$.

Theorem (4.4, Simplicity Criterion)

Let $\lambda \in \mathfrak{h}^*$. Then $M(\lambda) = L(\lambda)$ if and only if λ is antidominant.

Proof:

- Suppose $M(\lambda)$ is simple and that λ is not antidominant.
  - Then, we can find $\alpha > 0$ such that $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^{>0}$.
  - Since $\lambda - s_\alpha \cdot \lambda = \langle \lambda + \rho, \alpha^\vee \rangle \alpha$, then $s_\alpha \cdot \lambda < \lambda$.
  - So there exist a proper embedding $M(s_\alpha \cdot \lambda) \subset M(\lambda)$.
  - This contradicts the simplicity of $M(\lambda)$.
- Suppose λ is antidominant.
  - Then by (3.5), $\lambda \leq w \cdot \lambda$, for any $w \in W_{[\lambda]}$.
  - But $M(\lambda)$ only has composition factors $L(w \cdot \lambda)$ with $w \cdot \lambda \leq \lambda$ and $w \in W_{[\lambda]}$.
  - Therefore $L(\lambda)$ is the only composition factor and it occurs only once, so $M(\lambda) = L(\lambda)$. □

Simplicity Criterion: General Case

Corollary (4.8)

Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then for all $w \in W_{[\lambda]}$, the socle of $P(w \cdot \lambda)$ is a direct sum of copies of $L(\lambda)$.

Proof:

- Construct a standard filtration

$$0 = P_0 \subset P_1 \subset \cdots \subset P_n = P(w \cdot \lambda),$$

  with $P_i / P_{i-1} \cong M(w' \cdot \lambda)$ for some $w' \in W$.
- Take simple summand $L \subset \operatorname{Soc} P(w \cdot \lambda)$.
- Let i be the least index such that $L \subset P_i$, then $L \cap P_{i-1} = 0$.
- Then $L \subset M(w' \cdot \lambda) \cong P_i / P_{i-1}$.
- Then L is a Verma module with antidominant highest weight linked to $w' \cdot \lambda$, so $L \cong L(\lambda)$. □

Simplicity Criterion: General Case

Notes:

- It is hard to determine for what r of appear in $\operatorname{Soc} P(w \cdot \lambda) \cong L(\lambda)^r$.
- It can be shown (13.14), using Requires Kazhdan–Lusztig theory (8.4), that

$$r = (P(w \cdot \lambda) : M(w_\circ \cdot \lambda)) = [M(w_\circ \cdot \lambda) : L(w \cdot \lambda)].$$

Exercise:

- Let $\lambda \in \mathfrak{h}^*$. If $P(\lambda) \cong P(\lambda)^\vee$ is self-dual, i.e. $P(\lambda) \cong Q(\lambda)$. Then λ is antidominant.
  - What can we say about the converse? See Theorem 4.10.
- Solution:
  - $P(\lambda)$ has submodule $L(\mu)$, where μ is antidominant.
  - $P(\lambda) = Q(\lambda)$ is injective and indecomposable, so $Q(\lambda) \cong Q(\mu)$. Therefore, $\lambda = \mu$.

Blocks of $\mathcal{O}$ Revisited

Definition of blocks:

- Simple modules M_1 and M_2 are in the same block if $\text{Ext}_{\mathcal{O}}(M_1, M_2) \neq 0$ or $\text{Ext}_{\mathcal{O}}(M_2, M_1) \neq 0$.
- Two modules are in the same block if all their composition factors are.

Theorem (4.9)

The blocks of $\mathcal{O}$ are precisely the subcategories consisting of modules whose composition factors all have highest weights linked by $W_{[\lambda]}$ to an antidominant weight λ. Thus the blocks are in natural bijection with antidominant (or alternatively, dominant) weights.

We denote the individual blocks by $\mathcal{O}_\lambda$, where λ is antidominant.

Blocks of $\mathcal{O}$ Revisited

Proof:

- Enough to prove it for simple modules.
- Let $\mu \in \mathfrak{h}^*$. Then $M(\mu)$ has unique simple submodule $L(\lambda) = M(\lambda)$. Where λ is antidominant by Theorem 4.4.
- All composition factors of $M(\mu)$, including $L(\mu)$, are in the same block as $L(\lambda)$. Moreover, $\mu = w \cdot \lambda$ for some $w \in W_{[\lambda]}$.
- Furthermore, $L(\lambda)$ is the unique simple submodule of $M(w \cdot \lambda)$ for any $w \in W_{[\lambda]}$. So $L(w \cdot \lambda)$ is in the same block as $L(\lambda)$.
- Finally, suppose λ and λ' are both antidominant and $\lambda \neq \lambda'$. Then by Theorem 3.3 and Theorem 4.4

$$\begin{aligned} \mathrm{Ext}_{\mathcal{O}}(L(\lambda), L(\lambda')) &= \mathrm{Ext}_{\mathcal{O}}(L(\lambda), L(\lambda')^{\vee}) \\ &= \mathrm{Ext}_{\mathcal{O}}(M(\lambda), M(\lambda')^{\vee}) = 0 \end{aligned}$$

- Therefore, $L(\lambda)$ and $L(\lambda')$ are not in the same block. $\square$

Blocks of $\mathcal{O}$ Revisited

Notes:

- Suppose λ and λ' are antidominant and that $\mathcal{O}_\lambda, \mathcal{O}_{\lambda'} \subset \mathcal{O}_\chi$, for some central character χ.
- Then $|W_{[\lambda]} \cdot \lambda| = |W_{[\lambda']} \cdot \lambda'|$ and $\mathcal{O}_\lambda \cong \mathcal{O}_{\lambda'}$.
- Last part is not proven in this book.

Exercise:

- Suppose $M \in \mathcal{O}$ has a contravariant form, then its block summands in distinct blocks $\mathcal{O}_\lambda, \mathcal{O}_\mu$ are orthogonal.

Example: Antidominant Projectives

- If $\lambda + \rho \in \Lambda^+$, then λ is dominant and integral. and $w_\circ \cdot \lambda$ is antidominant and integral.

Theorem (4.10)

Let $\lambda + \rho \in \Lambda^+$. Then $P(w_\circ \cdot \lambda) \cong P(w_\circ \cdot \lambda)^\vee$ and $(P(w_\circ \cdot \lambda) : M(w \cdot \lambda)) = [M(w \cdot \lambda) : L(w_\circ \cdot \lambda)] = 1$ for all $w \in W$.

Proof:

- Consider the module $P(-\rho) = M(-\rho) = L(-\rho) = Q(-\rho)$.
  - This is projective and injective.
- Define the module $T := M(-\rho) \otimes L(\lambda + \rho)$.
  - Since $\dim L(\lambda + \rho) < \infty$, then T is projective and injective.
  - T has standard filtration $M(\mu - \rho)$, where μ runs over the weights of $L(\lambda + \rho)$.
  - $M(\mu - \rho)$ appears $\dim L(\lambda + \rho)_\mu$ times.
  - Each direct summand of T satisfies similar properties.

Example: Antidominant Projectives

Proof (continued):

- Consider the central character $\chi = \chi_\lambda$ and the block summand T^χ of T.
  - T^χ is projective and injective.
  - T^χ has standard filtration by $M(\mu - \rho)$'s, where μ runs over the weights of $L(\lambda + \rho)$, for which $\mu - \rho$ is linked to λ.
  - $M(\mu - \rho)$ appears $\dim L(\lambda + \rho)_\mu$ times.
  - If $\mu - \rho = w \cdot \lambda$, then $\mu = w \cdot (\lambda + \rho)$.
  - Since $\dim L(\lambda + \rho) < \infty$, then $\dim L(\lambda + \rho)_{w \cdot (\lambda + \rho)} = 1$.
- T^χ has standard filtration by $M(w \cdot \lambda)$'s, for $w \in W$, each occuring exactly once.
- In particular, T^χ has the $M(w_\circ \cdot \lambda) = L(w_\circ \cdot \lambda)$ as quotient.
- Now T^χ is projective, so it has $P(w_\circ \cdot \lambda)$ as direct summand.
  - Therefore, $(P(w_\circ \cdot \lambda) : M(w \cdot \lambda)) \leq 1$.
  - But $L(w_\circ \cdot \lambda)$ is the unique simple submodule of $M(w \cdot \lambda)$, so $[M(w \cdot \lambda) : L(w_\circ \cdot \lambda)] \geq 1$.
- So $(P(w_\circ \cdot \lambda) : M(w \cdot \lambda)) = 1$ and thus $T^\chi = P(w_\circ \cdot \lambda)$.

Example: Antidominant Projectives

Proof (continued):

- $w_\circ \cdot \lambda$ is antidominant, so the socle of $P(w_\circ \cdot \lambda)$ is $L(w_\circ \cdot \lambda)^r$, for some r.
- So $L(w_\circ \cdot \lambda)$ is a submodule of $P(w_\circ \cdot \lambda)$.
- Now $T^\chi = P(w_\circ \cdot \lambda)$ is injective and indecomposable, so $P(w_\circ \cdot \lambda)$ is the injective envelope of $L(w_\circ \cdot \lambda)$.
- In other words, $P(w_\circ \cdot \lambda) \cong Q(w_\circ \cdot \lambda) = P(w_\circ \cdot \lambda)^\vee$. □

Example: Antidominant Projectives

Notes:

- This theorem generalizes (7.16):
- Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then $P(\lambda) \cong P(\lambda)^\vee$ and $(P(\lambda) : M(w \cdot \lambda)) = [M(w \cdot \lambda) : L(\lambda)] = 1$ for all $w \in W_{[\lambda]}$.
- By Exercise 4.8, $P(\lambda) \cong P(\lambda)^\vee$ only when λ is antidominant.

Exercise:

- What can we say about $\dim \operatorname{End}_{\mathcal{O}} P(w_\circ \cdot \lambda)$?
- Solution: $\dim \operatorname{End}_{\mathcal{O}} P(w_\circ \cdot \lambda) = 1$?

Application to $\mathfrak{sl}(3,\mathbb{C})$

What are the composition factors of $M(w \cdot \lambda)$, when $\lambda \in \Lambda$ is antidominant and regular?

- For $\mathfrak{sl}(3,\mathbb{C})$, $\Delta = \{\alpha, \beta\}$ and $W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, w_\circ\}$.
- Linkage class: $\{\lambda, s_\alpha \cdot \lambda, s_\beta \cdot \lambda, s_\alpha s_\beta \cdot \lambda, s_\beta s_\alpha \cdot \lambda, w_\circ \cdot \lambda\}$.
- Composition factors of $M(\lambda)$:
  - $[M(\lambda) : L(\lambda)] = 1$
  - $[M(\lambda) : L(w \cdot \lambda)] = 0$ for $w \neq 1$.
  - $\operatorname{ch} L(\lambda) = \operatorname{ch} M(\lambda)$
- Composition factors of $M(s_\alpha \cdot \lambda)$:
  - $[M(s_\alpha \cdot \lambda) : L(s_\alpha \cdot \lambda)] = 1$
  - $[M(s_\alpha \cdot \lambda) : L(\lambda)] = 1$
  - $[M(s_\alpha \cdot \lambda) : L(w \cdot \lambda)] = 0$ for $w \notin \{1, s_\alpha\}$.
  - $\operatorname{ch} L(s_\alpha \cdot \lambda) = \operatorname{ch} M(s_\alpha \cdot \lambda) - \operatorname{ch} M(\lambda)$
- Remaining cases ($w \in \{s_\alpha s_\beta, s_\beta s_\alpha, w_\circ\}$): Section 5.4.
- General solution: Chapter 8.

Application to $\mathfrak{sl}(3, \mathbb{C})$

Exercise:

- Suppose $\lambda \in \Lambda$ is antidominant and in the α-hyperplane.
  - $\langle \lambda, \alpha^\vee \rangle = 0$.
  - Linkage class: $\{\lambda, s_\beta \cdot \lambda, s_\alpha s_\beta \cdot \lambda\}$.
  - $w_\circ \cdot \lambda = s_\alpha s_\beta \cdot \lambda$ is dominant.
  - $\lambda < s_\beta \cdot \lambda < s_\alpha s_\beta \cdot \lambda$.
- The composition factors of $M(\lambda)$ and $M(s_\beta \cdot \lambda)$ are know.
- Composition factors of $M(s_\alpha s_\beta \cdot \lambda)$:
  - $[M(s_\alpha s_\beta \cdot \lambda) : L(s_\alpha s_\beta \cdot \lambda)] = 1$
  - $[M(s_\alpha s_\beta \cdot \lambda) : L(\lambda)] = 1$
  - $[M(s_\alpha s_\beta \cdot \lambda) : L(s_\beta \cdot \lambda)] = r > 0$.
  - $\operatorname{ch} L(s_\alpha s_\beta \cdot \lambda) = \operatorname{ch} M(s_\alpha s_\beta \cdot \lambda) - r \operatorname{ch} M(s_\beta \cdot \lambda) + (r-1) \operatorname{ch} M(\lambda)$.
- Can we determine r?

Shapovalov Elements

Can we construct an embedding $M(s_\gamma \cdot \lambda) \subset M(\lambda)$ explicitly?

- Here $\lambda \in \mathfrak{h}^*$, $\gamma \in \Phi^+$ and $\langle \lambda + \rho, \gamma^\vee \rangle \geq 0$.
- $v^+ \in M(\lambda)$ maximal vector of weight λ.
- $\bar{v}^+ \in M(s_\gamma \cdot \lambda)$ maximal vector of weight $s_\gamma \cdot \lambda$.
- There is a unique (up to a scalar) $u \in U(\mathfrak{n}^-)$ such that
  - $u \cdot v^+$ is a maximal vector of weight $s_\gamma \cdot \lambda = \lambda - \langle \lambda + \rho, \gamma^\vee \rangle \gamma$.
  - Then embedding is given by $u' \cdot \bar{v}^+ \mapsto u' u \cdot v^+$.

How does u depend on λ?

- Hard to answer.
- Find instead element $\theta_{\gamma,r} \in U(\mathfrak{b}^-)_{-r\gamma}$, for $r > 0$, such that
  - $\theta_{\gamma,r} \in U(\mathfrak{b}^-)_{-r\gamma}$ is independent of λ.
  - $\theta_{\gamma,r} \cdot v^+$ is a maximal vector of weight $\lambda - r\gamma$ whenever $\langle \lambda + \rho, \gamma^\vee \rangle = r$ and v^+ is a maximal vector of weight λ.
- $\theta_{\gamma,r}$ is the Shapovalov element.

Shapovalov Elements

Example/Exercise:

- $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, $\Delta = \{\alpha, \beta\}$ and $\Phi^+ = \{\alpha, \beta, \gamma = \alpha + \beta\}$.
- Since α, β are simple, then $\theta_{\alpha,r} = y_\alpha^r$ and $\theta_{\beta,r} = y_\beta^r$.
- To determine $\theta_{\gamma,r}$ is difficult. We do only $\theta_{\gamma,1}$.
- We construct first the element $u \in U(\mathfrak{n}^-)_{-\gamma}$ dependent on λ.
  - Since $u \neq 0$, then $u = r y_\alpha y_\beta + s y_\gamma$, with r, s not both 0.
  - Write $\lambda = a\overline{\omega}_\alpha + b\overline{\omega}_\beta$.
  - Assume $\langle \lambda + \rho, \gamma^\vee \rangle = 1$, then $a + b = -1$.
  - Define hyperplane $H := \{\lambda = a\overline{\omega}_\alpha + b\overline{\omega}_\beta : a + b = -1\}$.
- If $u \cdot v^+ \in M(\lambda)$ is maximal vector of weight $\lambda - \gamma$, then
  - $0 = x_\alpha u \cdot v^+ = (r(a+1) - s) y_\beta \cdot v^+$.
  - $0 = x_\beta u \cdot v^+ = (rb + s) y_\alpha \cdot v^+$.
- This determines u in terms of $\lambda \in H$.
  - In all cases, $r \neq 0$.
  - u is unique up to a scalar
  - So we may take $r = 1$ and $s = -b = \lambda(h_\beta)$.

Shapovalov Elements

Example/Exercise:

- Now consider $y_\alpha y_\beta - y_\gamma h_\beta \in U(\mathfrak{b}^-)_{-\gamma}$.
- For all $\lambda \in H$ an maximal vector v^+ of weight λ:
  - $x_\alpha(y_\alpha y_\beta - y_\gamma h_\beta) \cdot v^+ = (a + 1 + b)y_\beta \cdot v^+ = 0$
  - $x_\beta(y_\alpha y_\beta - y_\gamma h_\beta) \cdot v^+ = (b - b)y_\alpha \cdot v^+ = 0.$
- So $\theta_{\gamma,1} = y_\alpha y_\beta - y_\gamma h_\beta$.
  - Clearly $\theta_{\gamma,1}$ is independent of λ.

Shapovalov Elements

Write $\Phi^+ = \{\alpha_1, \ldots, \alpha_m\}$ with $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Let $y_i \in U(\mathfrak{n}^-)$ correspond to $-\alpha_i$, for $i = 1, \ldots, m$.

Theorem (4.12, Shapovalov)

Fix $\gamma \in \Phi^+$ and an integer $r > 0$. There exists an element $\theta_{\gamma,r} \in U(\mathfrak{b}^-)_{-r\gamma}$ having the following properties:

a) *For each root $\beta > 0$, the commutator $[x_\beta, \theta_{\gamma,r}]$ lies in the left ideal $I_{\gamma,r} := U(\mathfrak{g})(h_\gamma + \rho(h_\gamma) - r) + U(\mathfrak{g})\mathfrak{n}$.*

b) *If $\gamma = \sum_{i=1}^{\ell} a_i\alpha_i$, then we can write*

$$\theta_{\gamma,r} = \prod_{i=1}^{\ell} y_i^{ra_i} + \sum_j p_j q_j,$$

with $p_j \in U(\mathfrak{n}^-)_{-r\gamma}$, $q_j \in U(\mathfrak{h})$, and $\deg p_j < r\sum_{i=1}^{\ell} a_i$.

Moreover, $\theta_{\gamma,r}$ is unique (up to a scalar) modulo the left ideal $J_{\gamma,r} := U(\mathfrak{b}^-)(h_\gamma + \rho(h_\gamma) - r)$.

Notes:

- Consider $\lambda \in \mathfrak{h}^*$, $\gamma \in \Phi^+$ and $r > 0$. Let $v^+ \in M(\lambda)$ be a maximal vector of weight λ.
- Then $\theta_{\gamma,r} \cdot v^+$ is a maximal vector of weight $\lambda - r\gamma = s_\gamma \cdot \lambda$, whenever $r = \langle \lambda + \rho, \gamma^\vee \rangle$.
- If so, then consider q_j from (b) as polynomial functions on $\mathfrak{h}^*$ and write

$$\theta_{\gamma,r}(\lambda) = \prod_{i=1}^{\ell} y_i^{ra_i} + \sum_j p_j q_j(\lambda).$$

- Then $\theta_{\gamma,r}(\lambda) \in U(\mathfrak{n}^-)_{-r\gamma}$ and $\theta_{\gamma,r}(\lambda) \cdot v^+$ is a maximal vector of weight $\lambda - r\gamma = s_\gamma \cdot \lambda$.
  - $\theta_{\gamma,r}(\lambda)$ is unique (up to a scalar) in $U(\mathfrak{n}^-)$.
- Difficult to construct $\theta_{\gamma,r}$ explicitly. We use a more round about approach.

Proof of Shapovalov's Theorem

Proof (Set-up, Induction in $\operatorname{ht}\gamma$):

- Fix $r > 0$ and $\gamma \in \Phi^+$.
  - If γ is simple ($\operatorname{ht}\gamma = 1$), then $\theta_{\gamma,r} = y_\gamma^r$.
  - This is our induction base.
- If $\gamma \notin \Delta$, then there exists $\alpha \in \Delta$ (0.2) such that
  - $p := \langle \gamma, \alpha^\vee \rangle > 0$.
  - $\beta := s_\alpha \gamma = \gamma - p\alpha > 0$, and $\operatorname{ht}\beta < \operatorname{ht}\gamma$.
- The induction hypothesis provides $\theta_{\beta,r} \in U(\mathfrak{b}^-)_{-r\beta}$ with the desired properties.
- Before applying this we discuss the proof strategy.

Proof of Shapovalov's Theorem

Proof (Strategy):

- Consider hyperplane $H_{\gamma,r}$ and half-space H_α:

$$H_{\gamma,r} := \{\lambda \in \mathfrak{h}^* : \langle \lambda + \rho, \gamma^\vee \rangle = r\}$$
$$H_\alpha := \{\lambda \in \mathfrak{h}^* : \langle \lambda + \rho, \alpha^\vee \rangle < 0\}$$

- $H_{\gamma,r}$ is Zariski closed.
- $\Lambda \cap H_{\gamma,r}$ is Zariski dense in $H_{\gamma,r}$.
- $\Theta := H_\alpha \cap \Lambda \cap H_{\gamma,r}$ is also Zariski dense in $H_{\gamma,r}$. Why?

Note that $H_{\gamma,r}$ contains exactly the weights λ for which we expect $\theta_{\gamma,r}$ to describe the embedding $M(s_\gamma \cdot \lambda) \hookrightarrow M(\lambda)$.

Proof of Shapovalov's Theorem

Proof (Strategy):

- Recall that I is the left ideal in $U(\mathfrak{g})$ annihilating any maximal vector $v^+ \in M(\lambda)$ of weight λ.
- Suppose we have $\theta_{\gamma,r}(\lambda) \in U(\mathfrak{n}^-)_{-r\gamma}$, for each $\lambda \in \Theta$, s.t.
  - a') $[x_\gamma, \theta_{\gamma,r}(\lambda)] \in I$.
  - b') Independent of the choice of $\lambda \in \Theta$, the highest degree term in $\theta_{\gamma,r}(\lambda)$ when written in a standard PBW basis is $\prod_i y_i^{ra_i}$.
  - c') The coefficients of $\theta_{\gamma,r}(\lambda)$ in the PBW basis depend polynomially on λ.
- By (c') there exists $p_j \in U(\mathfrak{n}^-)_{-r\gamma}$ and $q_j \in U(\mathfrak{h})$ such that $\theta_{\gamma,r}(\lambda) = \sum_j p_j q_j(\lambda)$.
- $\theta_{\gamma,r}(\lambda)$ can be extended to all $\lambda \in H_{\gamma,r}$:
  - Rewrite (a')-(c') as polynomial equations whose mutual solution space is then Zariski closed. It is then $H_{\gamma,r}$.
- Define then $\theta_{\gamma,r} := \sum_j p_j q_j \in U(\mathfrak{b}^-)_{-r\gamma}$.

Proof of Shapovalov's Theorem

Proof (Strategy):
The element $\theta_{\gamma,r} := \sum_j p_j q_j \in U(\mathfrak{b}^-)_{-r\gamma}$ then satisfies the conditions of the theorem.

a) Follows from (a'). How?

b) Follows directly from (b') and the definition of $\theta_{\gamma,r}$.

*) $\theta_{\gamma,r}$ is unique (up to a scalar) modulo $J_{\gamma,r}$.

- At each $\lambda \in H_{\gamma,r}$, $J_{\gamma,r}$ specializes to the annihilator in $U(\mathfrak{b}^-)$ any maximal vector $v^+ \in M(\lambda)$ of weight λ.
- At each $\lambda \in H_{\gamma,r}$, $\theta_{\gamma,r}$ specializes to $\theta_{\gamma,r}(\lambda)$, which is the unique (up to a scalar) element in $U(\mathfrak{n}^-)$ inducing $M(s_\gamma \cdot \lambda) \hookrightarrow M(\lambda)$.

Proof of Shapovalov's Theorem

Proof (Construction of $\theta_{\gamma,r}(\lambda)$, $\lambda \in \Theta$):

- Recall the data we have:
  - $r > 0$, $\gamma > 0$ with $\operatorname{ht}\gamma > 1$.
  - $\alpha \in \Delta$ with $p = \langle \gamma, \alpha^\vee \rangle > 0$, $\beta := s_\alpha \gamma = \gamma - p\alpha > 0$ and $\operatorname{ht}\beta < \operatorname{ht}\gamma$.
  - $\theta_{\beta,r} \in U(\mathfrak{b}^-)_{-r\beta}$, satisfying the conditions of the theorem.
  - $\theta_{\beta,r}(\mu)$, for all $\mu \in H_{\beta,r}$ satisfying (a')-(c').
- For $\lambda \in \Theta$ there is $q \in \mathbb{Z}^{>0}$ such that
  - $\mu := s_\alpha \cdot \lambda = \lambda + q\alpha > \lambda$, $\mu \in H_{\beta,r}$.
  - $s_\alpha \cdot (\mu - r\beta) = \lambda - r\gamma$, with $\langle \mu - r\beta + \rho, \alpha^\vee \rangle = q + rp$.
- Writing $n := q + rp > 0$ we get embeddings:
  - $M(\lambda) \hookrightarrow M(\mu)$, induced by y_α^q.
  - $M(\mu - r\beta) \hookrightarrow M(\mu)$, induced by $\theta_{\beta,r}(\mu)$.
  - $M(\lambda - r\gamma) \hookrightarrow M(\mu - r\beta)$, induced by y_α^n.

Proof of Shapovalov's Theorem

Proof (Construction of $\theta_{\gamma,r}(\lambda)$, $\lambda \in \Theta$):

- We have the embeddings:
  - $M(\lambda) \hookrightarrow M(\mu)$, induced by y_α^q.
  - $M(\mu - r\beta) \hookrightarrow M(\mu)$, induced by $\theta_{\beta,r}(\mu)$.
  - $M(\lambda - r\gamma) \hookrightarrow M(\mu - r\beta)$, induced by y_α^n.
  - y_α^q, $\theta_{\beta,r}(\mu)$ and y_α^n are unique in $U(\mathfrak{n}^-)$ up to a scalar.
- Furthermore the embedding $M(\lambda - r\gamma) \hookrightarrow M(\lambda)$ is induced by a unique element $\theta_{\gamma,r}(\lambda)$, satisfying:

$$\theta_{\gamma,r}(\lambda) y_\alpha^q = y_\alpha^n \theta_{\beta,r}(\mu)$$

- $\theta_{\gamma,r}(\lambda)$ then satisfies the properties (a')-(c').
  - a') $[x_\gamma, \theta_{\gamma,r}(\lambda)] \in I$. Why?
  - b') Follows by comparing highest degree terms on either side.
  - c') Pull all y_α to the right on both sides and remove y_α^n. Rewriting into PBW no extra dependencies on λ appear. (c') then follows since $\theta_{\beta,r}$ satisfy (c') and $\mu = \lambda + r\alpha$. □