Content

In this chapter we will answer the following questions:

1) What are the simple submodules of $M(\lambda)$?
2) When is $M(\lambda)$ simple?
3) When does an embedding $M(\mu) \to M(\lambda)$ exist?
4) Can we construct such an embedding explicitly?
5) What are the blocks of $\mathcal{O}$?

Today: (1), (2) for $\lambda \in \Lambda$ and (3) for $\mu, \lambda \in \Lambda$.

Simple Submodules of Verma Modules

Lemma (4.1)

Any two nonzero left ideals of a left noetherian ring without zero divisors must intersect nontrivially.

Proposition (4.1)

For any $\lambda \in \mathfrak{h}^*$, the module $M(\lambda)$ has a unique simple submodule, which is therefore its socle.

Proof:

- $M(\lambda)$ is artinian, so it has a simple submodule.
- Suppose L, L' are distinct simple submodule of $M(\lambda)$, then $L \cap L' = \{0\}$.
- As $U(\mathfrak{n}^-)$-modules, $M(\lambda) \cong U(\mathfrak{n}^-)$. So L and L' are left ideals of $U(\mathfrak{n}^-)$.
- $U(\mathfrak{n}^-)$ is a left noetherian ring without zero divisors, so $L \cap L' \neq \{0\}$. This is a contradiction. □

Simple Submodules of Verma Modules

Note:

- The simple submodule of $M(\lambda)$ is isomorphic to some $L(\mu)$ with $\mu \leq \lambda$. Moreover, $\mu = w \cdot \lambda$ for some $w \in W_{[\lambda]}$.

Exercise:

- Let M be a nonzero submodule of $M(\lambda)$. Then M has a nondegenerate contravariant form if and only if it is the unique simple submodule of $M(\lambda)$.

Homomorphisms Between Verma Modules

Theorem (4.2)

Let $\lambda, \mu \in \mathfrak{h}^*$.

a) *Any nonzero homomorphism $\varphi : M(\mu) \to M(\lambda)$ is injective.*
b) *In all cases,* $\dim \operatorname{Hom}(M(\mu), M(\lambda)) \leq 1$.
c) *The unique simple submodule $L(\mu)$ in $M(\lambda)$ is a Verma module.*

Proof:

a) As a $U(\mathfrak{n}^-)$-module homomorphism $\varphi : U(\mathfrak{n}^-) \to U(\mathfrak{n}^-)$, $u' \mapsto u'u$, for some fixed $u \neq 0$.
Since $U(\mathfrak{n}^-)$ has no zero-divisors, $\operatorname{Ker} \varphi = 0$.
b) Consider nonzero $\varphi_1, \varphi_2 : M(\mu) \to M(\lambda)$ and unique simple submodule $L \subset M(\mu)$. Then $\varphi_1(L) = \varphi_2(L)$ is simple.
Let $\varphi_3 = \varphi_1 - c\varphi_2$ s.t. $\varphi_3(L) = \{0\}$. Then $\varphi_3 = 0$ by (a).
c) By universal property of $M(\mu)$, there exists $\varphi : M(\mu) \to M(\lambda)$, with $\varphi(M(\mu)) = L(\mu)$. Now φ is injective by (a). □

Homomorphisms Between Verma Modules

Notes:

- Whenever $\text{Hom}(M(\mu), M(\lambda)) \neq 0$ we can now unambiguously write $M(\mu) \subset M(\lambda)$.
- One major goal in this chapter is to study this embedding.
  - When does it exist?
  - How can we construct it?
- The other goal is to determine when $M(\lambda)$ is simple.

Special Case: λ is a Dominant Integral Weight

Proposition (4.3)

Suppose $\lambda + \rho \in \Lambda^+$. Then $M(w \cdot \lambda) \subset M(\lambda)$, for all $w \in W$; thus all $[M(\lambda) : L(w \cdot \lambda)] > 0$.
More precisely, if w has reduced expression $w = s_n \cdots s_1$, with s_i reflection relative to the simple root α_i, then there is a sequence

$$M(w \cdot \lambda) = M(\lambda_n) \subset M(\lambda_{n-1}) \subset \cdots \subset M(\lambda_0) = M(\lambda),$$

where $\lambda_0 := \lambda$ and $\lambda_k := s_k \cdot \lambda_{k-1}$, for $k \in \{1, \ldots, n\}$.
In particular, $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_0$, with $\langle \lambda_k + \rho, \alpha_{k+1}^\vee \rangle \in \mathbb{Z}^+$, for $k = 0, \ldots, n-1$.

Proof:
Use the reformulated Proposition 1.4 and Theorem 4.2(c):

- If $\langle \lambda_k + \rho, \alpha_{k+1}^\vee \rangle \in \mathbb{Z}^+$, then there exists an embedding $M(s_{k+1} \cdot \lambda_k) = M(\lambda_{k+1}) \to M(\lambda_k)$. $\square$

Special Case: λ is a Dominant Integral Weight

Exercises: Assume $\lambda + \rho \in \Lambda^+$.

- The unique simple submodule of $M(\lambda)$ is isomorphic to $M(w_\circ \cdot \lambda)$.
- If $\lambda \in \Lambda^+$, then all inclusions in the proposition are proper.

Notes:

- This proposition will generalize as follow:

  Let $\lambda \in \mathfrak{h}^*$. Given $\alpha > 0$, suppose $\mu := s_\alpha \cdot \lambda \leq \lambda$. Then there exists an embedding $M(\mu) \subset M(\lambda)$.

- The failure of Proposition 1.4 to carry over to nonsimple positive roots, means that a totally different strategy is needed to generalize this result.

Simplicity Criterion: Integral Case

Recall that $\lambda \in \mathfrak{h}^*$ is called antidominant if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ for all $\alpha \in \Phi^+$.

Theorem (Simplicity Criterion)

Let $\lambda \in \mathfrak{h}^*$. Then $M(\lambda) = L(\lambda)$ if and only if λ is antidominant.

Proof (Integral case, $\lambda \in \Lambda$):

- Suppose $M(\lambda)$ is simple and that λ is not antidominant.
  - Then, since $\lambda \in \Lambda$, we can find a simple root α such that $\langle \lambda + \rho, \alpha^\vee \rangle > 0$.
  - Using the Proposition 1.4 and Theorem 4.2(c), we get a proper embedding $M(s_\alpha \cdot \lambda) \to M(\lambda)$.
  - This contradicts the simplicity of $M(\lambda)$.
- Suppose λ is antidominant.
  - Then by (3.5), $\lambda \leq w \cdot \lambda$.
  - But $M(\lambda)$ only has composition factors $L(w \cdot \lambda)$ with $w \cdot \lambda \leq \lambda$.
  - Therefore $L(\lambda)$ is the only composition factor and it occurs only once, so $M(\lambda) = L(\lambda)$. □

Simplicity Criterion: Integral Case

Exercise:

- If $\lambda \in \Lambda$ is antidominant, then the socle of $P(w \cdot \lambda)$ with $w \in W$ is a direct sum of copies of $L(\lambda)$.
- The general version of this result is proven later.

Notes:

- Only the second part of this proof can be generalized to $\lambda \in \mathfrak{h}^*$.
- The first part does not generalize since Proposition 1.4 no longer applies.
- We therefore need more information on embeddings $M(s_\alpha \cdot \lambda) \subset M(\lambda)$, where α is not simple.

Existence of Embeddings: Preliminaries

Proposition (4.5)

Let $\mu, \lambda \in \mathfrak{h}^*$ *and* $\alpha \in \Delta$, *with* $n := \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}$ *and*

$$M(s_\alpha \cdot \mu) \subset M(\mu) \subset M(\lambda).$$

Then there are two possibilities for the position of $M(s_\alpha \cdot \lambda)$*:*

a) *If* $n \leq 0$, *then* $M(\lambda) \subset M(s_\alpha \cdot \lambda)$.

b) *If* $n > 0$, *then* $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda) \subset M(\lambda)$.

Lemma (4.5)

Let $\mathfrak{a}$ *be a nilpotent Lie algebra, with* $x \in \mathfrak{a}$ *and* $u \in U(\mathfrak{a})$. *Given a positive integer* n, *there exists an integer* t *depending on* x *and* u *such that* $x^t u \in U(\mathfrak{a}) x^n$.

We also note that for any $\alpha \in \Delta$ and $t > 0$,

$$[x, y_\alpha^t] = t y_\alpha^{t-1}(h_\alpha - t + 1).$$

Existence of Embeddings: Preliminaries

Proof (Proposition 4.5):

a) If $n \leq 0$, then $M(\lambda) \subset M(s_\alpha \cdot \lambda)$ by Proposition 1.4, since

$$\langle s_\alpha \cdot \lambda + \rho, \alpha^\vee \rangle = \langle s_\alpha \cdot (\lambda + \rho), \alpha^\vee \rangle = \langle \lambda + \rho, -\alpha^\vee \rangle = -n \geq 0.$$

b) If $n > 0$, then we want $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda) \subset M(\lambda)$.

- Proposition 1.4 immediately gives $M(s_\alpha \cdot \lambda) \subset M(\mu)$.
- Letting $s := \langle \mu + \rho, \alpha^\vee \rangle$ we get maximal vectors

$$v_\lambda^+ \in M(\lambda), \quad y_\alpha^n \cdot v_\lambda^+ \in M(s_\alpha \cdot \lambda), \quad v_\mu^+ \in M(\mu), \quad y_\alpha^s \cdot v_\mu^+ \in M(s_\alpha \cdot \mu).$$

- Since $M(\mu) \subset M(\lambda)$, there is $u \in U(\mathfrak{n}^-)$ with $v_\mu^+ = u \cdot v_\lambda^+$.
- Lemma 4.5 gives us $t \geq s$ such that $y_\alpha^t u \in U(\mathfrak{n}^-) y_\alpha^n$. So

$$y_\alpha^t \cdot v_\mu^+ = y_\alpha^t u \cdot v_\lambda^+ \in U(\mathfrak{n}^-) y_\alpha^n v_\lambda^+ \subset M(s_\alpha \cdot \lambda).$$

- If $t > s$, then we use $[x, y_\alpha^t] = t y_\alpha^{t-1}(h_\alpha - t + 1)$ to get

$$(s - t) t y_\alpha^{t-1} v_\mu^+ = x_\alpha y_\alpha^t \cdot v_\mu^+ \in M(s_\alpha \cdot \lambda).$$

- This proves $y_\alpha^s \cdot v_\mu^+ \in M(s_\alpha \cdot \lambda)$ and $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda)$. □

Existence of Embeddings: Integral Case

Theorem (4.6, Verma)

Let $\lambda \in \mathfrak{h}^*$. Given $\alpha > 0$, suppose $\mu := s_\alpha \cdot \lambda \leq \lambda$. Then there exists an embedding $M(\mu) \subset M(\lambda)$.

Proof (Integral case, $\lambda \in \Lambda$):

- Since λ is integral, so is μ. Therefore we can find $w \in W$ such that $\mu' := w^{-1} \cdot \mu \in \Lambda^+ - \rho$.
- Considering a reduced expression $w = s_n \cdots s_1$, we define weights $\mu_0 := \mu'$ and $\mu_k := s_k \cdot \mu_{k-1}$, for $k = 1, \ldots, n$.
- Proposition 4.3 tells us that $\mu_0 \geq \cdots \geq \mu_n$ and that

$$M(\mu_0) \supset M(\mu_1) \supset \cdots \supset M(\mu_n).$$

- Letting $\lambda' := w^{-1} \cdot \lambda$ we define a parallel list of weights. $\lambda_0 := \lambda'$ and $\lambda_k := s_k \cdot \lambda_{k-1}$, for $k = 1, \ldots, n$.

Existence of Embeddings: Integral Case

Proof (continued):

- A short calculation shows that if $w_k := s_{k+1} \cdots s_n$, then $\mu_k = s_{\beta_k} \cdot \lambda_k$, β_k is the root with $s_{\beta_k} = w_k^{-1} s_\alpha w_k$.
- It follows that $\mu_k - \lambda_k \in \mathbb{Z}\beta_k$.
- We may assume that $\mu < \lambda$. This implies $\mu_k \neq \lambda_k$ and in particular $\mu' > \lambda'$, since μ' is dominant.
- There must thus be a least index k such that $\mu_k > \lambda_k$ and $\mu_{k+1} < \lambda_{k+1}$. We fix this k.
- We will prove $M(\mu_{k+1}) \subset M(\lambda_{k+1})$, $M(\mu_{k+2}) \subset M(\lambda_{k+2})$,... Culminating in $M(\mu_n) \subset M(\lambda_n)$.
- By definition $\mu_{k+1} - \lambda_{k+1} = s_{k+1}(\mu_k - \lambda_k)$.
- By our choice of k, we get $\mu_{k+1} - \lambda_{k+1} \in \mathbb{Z}^- \beta_{k+1}$ and $s_{k+1}(\mu_k - \lambda_k) \in \mathbb{Z}^+ \beta_k$. So $\beta_k = \beta_{k+1} = \alpha_{k+1}$.
- Prop 1.4 yields $M(\mu_{k+1}) = M(s_{k+1} \cdot \lambda_{k+1}) \subset M(\lambda_{k+1})$.

Existence of Embeddings: Integral Case

Proof (continued):

- Combined with the sequence of embeddings we get

$$M(\mu_{k+2}) = M(s_{k+2} \cdot \mu_{k+1}) \subset M(\mu_{k+1}) \subset M(\lambda_{k+1}).$$

- Proposition 4.5 then implies that

$$M(\mu_{k+2}) \subset M(s_{k+2} \cdot \lambda_{k+1}) = M(\lambda_{k+2}).$$

- Iterating these last arguments we get

$$M(\mu) = M(\mu_n) \subset M(\lambda_n) = M(\lambda).$$

□

Extra: Solution to problem we discussed at the end

Unless I mistake something in the definition of the partial ordering. Let $\mu \in \mathfrak{h}^*$ and $\alpha \in \Delta$. Assume $M(s_\alpha \cdot \mu) \subset M(\mu)$. Then $s_\alpha \cdot \mu \leq \mu$ or in other words

$$\langle \mu - s_\alpha \cdot \mu, \beta^\vee \rangle \in 2\mathbb{Z}^+, \quad (\forall \beta \in \Delta).$$

Noting that

$$\langle \mu + \rho, \alpha^\vee \rangle = -\langle s_\alpha \cdot (\mu + \rho), \alpha^\vee \rangle = -\langle s_\alpha \cdot \mu + \rho, \alpha^\vee \rangle,$$

we get

$$\langle \mu + \rho, \alpha^\vee \rangle = \frac{1}{2} \langle \mu - s_\alpha \cdot \mu, \beta^\vee \rangle \in \mathbb{Z}^+.$$