Category $\mathcal{O}$: Methods

- The functors Hom and Ext
- A duality functor $M \mapsto M^\vee$ on $\mathcal{O}$
- Reflection groups, dominant and antidominant weights
- Tensoring Verma modules with finite dimensional modules
- “Standard” filtrations having Verma modules as subquotients
- Projective objects in $\mathcal{O}$ and BGG reciprocity
- “Contravariant” forms on modules

Indecomposable modules for $\mathfrak{sl}(2, \mathbb{C})$

Proposition

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $\{\lambda, \mu\}$ be a linkage class of integral weights with $\lambda \geq 0$, $\mu = -\lambda - 2$ and $\chi = \chi_\lambda$. Every indecomposable module in $\mathcal{O}_\chi$ is isomorphic to one of the following five modules.

$$L(\lambda), \quad L(\mu) = M(\mu), \quad M(\lambda) = P(\lambda), \quad M(\lambda)^\vee = Q(\lambda), \quad P(\mu) = Q(\mu).$$

Proof:

$$\cdots \to \operatorname{Hom}_{\mathcal{O}}(L(\lambda), M(\lambda)) \to \operatorname{Hom}_{\mathcal{O}}(M(\lambda), M(\lambda)) \to \operatorname{Hom}_{\mathcal{O}}(N(\lambda), M(\lambda)) \to$$

$$\to \operatorname{Ext}_{\mathcal{O}}(L(\lambda), M(\lambda)) \to \operatorname{Ext}_{\mathcal{O}}(M(\lambda), M(\lambda)) \to \cdots$$

$\operatorname{Hom}_{\mathcal{O}}(L(\lambda), M(\lambda)) = 0$, $\operatorname{Hom}_{\mathcal{O}}(M(\lambda), M(\lambda)) = \mathbb{C}$, $N(\lambda) = L(\mu)$, $\operatorname{Hom}(L(\mu), L(\mu)) = \mathbb{C}$, $\operatorname{Ext}_{\mathcal{O}}(M(\lambda), M(\lambda)) = 0$, $\operatorname{Ext}_{\mathcal{O}}(L(\lambda), M(\lambda)) = 0$ (□)

Proof

Suppose $M \in \mathcal{O}_\chi$ not $\cong$ [(*)] and indecomposable

$\Rightarrow$ Only possible factors are $L(\lambda), L(\mu)$ (Δ)

all nonzero $v \in M_\lambda$ are maximal $\Rightarrow v$ generates submodule $\cong L(\lambda)$ or $M(\lambda)$

($\square$) + Ext commutes with direct sums + Induction

$\Rightarrow M_\lambda$ generates submodule $N \subset M$, of the form

$$N = L(\lambda) \oplus \dots \oplus L(\lambda) \oplus M(\lambda) \oplus \dots \oplus M(\lambda)$$

$\Rightarrow$ N is proper submodule of $M \overset{(\Delta)}{\Rightarrow} M/N \cong L(\mu) \oplus \dots \oplus L(\mu)$

At least one $L(\mu)$ extends N nontrivially

$\Rightarrow L(\mu)$ extends a $L(\lambda)$ or $M(\lambda)$ nontrivially $\Rightarrow Q(\lambda)$ OR $Q(\mu) \subset M$

$\overset{\text{injective prop}}{\Rightarrow}$ M splits

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P is a projective generator of $\mathcal{O}_\chi$ and
$A := \operatorname{End}_\mathcal{O} P$ is a finite dimension algebra.

Property of Projective generators

Proposition

The functor $\mathrm{Hom}_{\mathcal{O}}(P, \cdot)$ *defines a category equivalence between* $\mathcal{O}_\chi$ *and the categroy of finite dimensional right* A*-modules.*

7. Contravariant forms

Definition

A symmetric bilinear form $(v, v')_M$ *on* $M \in \mathrm{Mod}_{U(\mathfrak{g})}$ *is* **contravariant** *if*

$$(u \cdot v, v')_M = (v, \tau(u) \cdot v')_M \quad \textit{for all } u \in U(\mathfrak{g}) \textit{ and } v, v' \in M.$$

$\mathfrak{sl}(2,\mathbb{C})$ example

$\mathfrak{h}^* = \mathbb{C}$

$$\begin{cases} h v_i = (\lambda - 2i) v_i \\ x v_i = (\lambda - i + 1) v_{i-1} \\ y v_i = (i+1) v_{i+1} \end{cases}$$

$$(h v_i, v_j) = (\lambda - 2i)(v_i, v_j)$$

$$\| \quad \| $$

$$(v_i, h v_j) = (\lambda - 2j)(v_i, v_j) \Rightarrow (v_i, v_j) = 0 \text{ if } i \neq j$$

$$(x v_i, v_{i-1}) = (\lambda - i + 1)(v_{i-1}, v_{i-1}) \Rightarrow (v_i, v_i) = \frac{\lambda - i + 1}{i}(v_{i-1}, v_{i-1})$$

$$\|$$

$$(v_i, y v_{i-1}) = i (v_i, v_i)$$

$$(-\lambda)_i := (-\lambda)(-\lambda+1)\cdots(-\lambda+i-1)$$

$$\Rightarrow (v_i, v_i) = \frac{(-1)^i}{i!}(-\lambda)_i (v_0, v_0), \quad (v_0, v_0) = \text{a nonzero scalar constant}$$

$$\lambda \in \mathbb{N}: \quad (v_i, v_i) = \binom{\lambda}{i}(v_0, v_0)$$

Properties of contravariant forms

(a) $(M_\lambda, M_\mu)_M = 0$ whenever $\lambda \neq \mu$.

(a) $\lambda(h)(v, v') = (h \cdot v, v') = (v, h \cdot v') = \mu(h)(v, v')$

b)

Properties of contravariant forms

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(b) If $M = U(\mathfrak{g}) \cdot v_\lambda^+$ has a nonzero contravariant form, then it is uniquely determined up to a scalar multiple by $(v_\lambda^+, v_\lambda^+)_M$

(b) w.l.o.g. $v = u \cdot v_\lambda^+$, $v' = u' v_\lambda^+ \in M_\mu$ with $u, u' \in U(\mathfrak{n}^-)$

$u, u' : M_\lambda \to M_\mu \Rightarrow \tau(u), \tau(u') : M_\mu \to M_\lambda$

$(v, v') = (u v_\lambda^+, u' v_\lambda^+) = (v_\lambda, \tau(u) u' v_\lambda^+) = c(v_\lambda, v_\lambda)$ for some $c \in \mathbb{C}$

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(c) $(v \otimes w, v' \otimes w')_M := (v, v')_{M_1}(w, w')_{M_2}$ is a (non-degenerate) contravariant form of $M := M_1 \otimes M_2$ if the forms on M_1 and M_2 are (non-degenerate) contravariant forms.

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(d) If N is a submodule of M, then the orthogonal space

$$N^\perp := \{v \in M | (v, v')_M = 0 \text{ for all } v' \in N\}$$

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(c)(d) Trivial

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(e) If $M \in \mathcal{O}$ has a contravariant form, then the M^χ for distinct central characters χ are orthogonal.

$M^\chi = \{v \in M : (z - \chi(z))^n \cdot v = 0$ for a $n > 0$ depending on $z\}$

Proof

(e) Let χ_1, χ_2 be distinct: TP $\forall \lambda \in \mathfrak{h}^*$: $(M_\lambda^{\chi_1}, M_\lambda^{\chi_2}) = 0$

Fix $z \in Z(\mathfrak{g})$ s.t. $\chi_1(z) = c_1 \neq c_2 = \chi_2(z)$

Suppose first $v, w \in M$ are eigenvectors of z

$c_1(v,w) = (z \cdot v, w) = (v, z \cdot w) = c_2(v,w)$

In general $\forall v_i \in M_\lambda^{\chi_i}$ $\exists$ minimal $e_i \in \mathbb{Z}^+$: $(z - c_i)^{e_i} v_i = 0$

Use induction on $e_1 + e_2$

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- $\phi := \epsilon^{-} \otimes \mathrm{id} \otimes \epsilon^{+} : U(\mathfrak{g}) \cong U(\mathfrak{n}^{-}) \oplus U(\mathfrak{h}) \oplus U(\mathfrak{n}^{+}) \to U(\mathfrak{h})$
- The **universal bilinear form** on $U(\mathfrak{g})$,

$$C(u, u') := \phi(\tau(u)u')$$

Properties of the Universal form

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- $C(u, u')$ *is symmetric.*
- $C(1, 1) = 1$*.*
- $C(u, u')$ *is "contravariant":*

$$C(u_0 u, u') = C(u, \tau(u_0)u')$$

for all $u_0, u, u' \in U(\mathfrak{g})$*.*

Induced $\mathbb{C}$-valued form

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The linear map

$$\phi_\lambda := \lambda \circ \phi : U(\mathfrak{g}) \to \mathbb{C}$$

defines a $\mathbb{C}$-valued symmetric bilinear form

$$C^\lambda(u, u') := \phi_\lambda(\tau(u)u')$$

on $U(\mathfrak{g})$.

Corollary of the Universal construction

Theorem

Let M be a highest weight module of weight λ generated by a maximal vector v^+.

There exists a (nonzero) contravariant form $(v, v')_M$ on M, which is uniquely determined up to scalar multiples by $(v^+, v^+)_M$.

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Its radical is the unique maximal submodule of M.

In particular, the form is nondegenerate if and only if $M \cong L(\lambda)$.

Proof

$M' = \sum_{\mu \neq \lambda} M_\mu$, $N \subset M'$ unique maximal submodule of M

Note that: For $u \in U(\mathfrak{g})$: $u \cdot v^+ \equiv \varphi_\lambda(u) v^+ \pmod{M'}$

$u \cdot v^+ \in N \Leftrightarrow U(\mathfrak{g}) u \cdot v^+ \subset N \Leftrightarrow \varphi_\lambda(U(\mathfrak{g}) u) = 0$

$\Leftrightarrow C^\lambda(U(\mathfrak{g}), u) = 0 \Leftrightarrow u$ is in radical of C^λ

In particular $\forall u_1, u_2 \in U(\mathfrak{n}^-)$: $u_1 v^+ = u_2 v^+ \Rightarrow u_1 - u_2$ in radical of C^λ

$\Rightarrow (v, v')_M := C^\lambda(u, u')$ with $u, u' \in U(\mathfrak{n}^-)$ s.t.

$v = u \cdot v^+$, $v' = u' v^+$ well defined

N is radical of $(\cdot,\cdot)_M$, $(\cdot,\cdot)_M$ inherits contravariance of C^λ