Category $\mathcal{O}$: Methods

- The functors Hom and Ext
- A duality functor $M \mapsto M^\vee$ on $\mathcal{O}$
- Reflection groups, dominant and antidominant weights
- Tensoring Verma modules with finite dimensional modules
- "Standard" filtrations having Verma modules as subquotients
- Projective objects in $\mathcal{O}$ and BGG reciprocity
- "Contravariant" forms on modules

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Definition

A category $\mathcal{C}$ has **enough projectives** *if for all $M \in \mathcal{C}$ there exists a projective $P \in \mathcal{C}$ and an epimorphism $P \to M$.*

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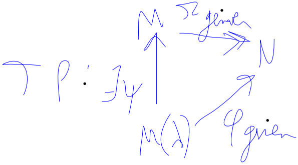
In $\mathcal{O}$ the existence of enough projectives implies the existence of enough injectives, thanks to the duality functor.

Therefore we just focus on projectives.

Enough Projectives in $\mathcal{O}$

Proposition

(a) *Suppose $\lambda \in \mathfrak{h}^*$ is dominant in $W_{[\lambda]} \cdot \lambda$. Then $M(\lambda)$ is projective in $\mathcal{O}$.*



Set $\chi = \chi_\lambda$. Since $M(\lambda) \in \mathcal{O}_\chi$ we have

$\mathrm{im}(\varphi) \subset N^\chi \Rightarrow$ may assume $M, N \in \mathcal{O}_\chi$

v^+ max vect in $M(\lambda)$ $\Rightarrow$ $\varphi(v^+)$ max vect of weight λ in N

π surjective $\Rightarrow$ $\exists\, v$ of weight λ in M s.t. $\varphi(v^+) = \pi(v)$

$M' :=$ π-submodule of generated by v must contain max vector of weight linked to λ $\overset{\lambda \text{ dominant}}{\Longrightarrow}$ $\pi \cdot v = 0$ $\underset{\text{universal prop of } M(\lambda)}{\Longrightarrow}$ $M(\lambda) \twoheadrightarrow M' : v^+ \mapsto v'$

Enough Projectives in $\mathcal{O}$

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(a) *Suppose $\lambda \in \mathfrak{h}^*$ is dominant in $W_{[\lambda]} \cdot \lambda$. Then $M(\lambda)$ is projective in $\mathcal{O}$.*

(b) *If $P \in \mathcal{O}$ is projective and $\dim L < \infty$, then $P \otimes L$ is projective in $\mathcal{O}$.*

TP: $\mathrm{Hom}(P \otimes L, \cdot)$ is exact

$L^* \otimes \cdot$ is exact

$\mathrm{Hom}_{\mathcal{O}}(P \otimes L, M) \cong \mathrm{Hom}_{\mathcal{O}}(P, \mathrm{Hom}(L, M)) \cong \mathrm{Hom}(P, L^* \otimes M)$

Pf

$\mathrm{Hom}(P, \cdot)$ exact because P is projective

Proof

Enough Projectives in $\mathcal{O}$

(*) Corollary of tensoring with verma modules

Theorem

Category $\mathcal{O}$ has enough projectives.

- First goal: $\forall \lambda \in \mathfrak{h}^*$ find a projective obj in $\mathcal{O}$ mapping onto $L(\lambda)$

$\mu := \lambda + n\rho$ is dominant for n large enough $\overset{(a)}{\Longrightarrow}$ $M(\mu)$ is projective

$n\rho \in \Lambda^+ \Rightarrow \dim(L(n\rho)) < \infty \overset{(b)}{\Longrightarrow} P := M(\mu) \otimes L(n\rho)$ is projective

lowest weight of $L(n\rho)$ is $-n\rho = w_0(n\rho) \overset{(*)}{\Longrightarrow}$

P has quotient $\cong M(\mu - n\rho) = M(\lambda) \Rightarrow M(\lambda)$ is quotient of a projective module

$\Rightarrow L(\lambda)$ is quotient of a projective module

Proof

- $M \in \mathcal{O}$. Induction on length of M

Assume $\ell(M) > 1 \Rightarrow \exists$ (SES) $0 \to L(\lambda) \to M \to N \to 0$

$\ell(N) < \ell(M) \overset{IH}{\Longrightarrow} \exists \varphi : Q \twoheadrightarrow N$ for some projective Q

Projective property $\Rightarrow$ $\exists$ lifting $\psi : Q \to M$.

Assume ψ is not surjective $\Rightarrow M \cong L(\lambda) \oplus N$

$\Rightarrow$ take direct sum of projective mapping onto $L(\lambda)$ and N

Remark

Corollary

Suppose $\lambda \in \mathfrak{h}^*$, $n \in \mathbb{N}$.
If $\mu := \lambda + n\rho \in \mathfrak{h}^*$ *is dominant, then the projective* $P := M(\mu) \otimes L(n\rho)$ *has standard filtration with subquotient* $M(\mu + \nu)$, *where* ν *runs over the weights of* $L(n\rho)$, *counting multiplicity.*

In particular, $M(\lambda)$ *occurs just once and* $\mu + \nu \geq \lambda$.

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$\pi : P_M \to M$ *is a* **projective cover** *of* $M \in C$ *if* π *is an essential epimorphism.*

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$\mathcal{O}$ is an artinian module category with enough projectives:

$$\text{All } M \in \mathcal{O} \text{ have a projective cover } \pi : P_M \to M$$

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$\pi : P_M \to M$ is a **projective cover** *of $M \in C$ if π is an essential epimorphism.* and P_M is projective

$\mathcal{O}$ is an artinian module category with enough projectives:

All $M \in \mathcal{O}$ have a projective cover $\pi : P_M \to M$

P_M is unique up to isomorphism.

Indecomposable projectives

For all $\lambda \in \mathfrak{h}^*$ we denote a fixed projective cover of $L(\lambda)$ by

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- $P(\lambda)$ *is indecomposable.*

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- *$P(\lambda)$ has $\ker \pi_\lambda$ as its unique maximal submodule.*
- *$P(\lambda)$ is indecomposable.*
- *$P(\lambda)$ is a projective cover of $M(\lambda)$.*

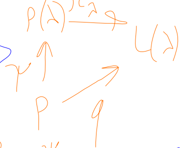
Properties of $P(\lambda)$

Theorem

(a) *In $\mathcal{O}$, every indecomposable projective module is isomorphic to a $P(\lambda)$*

$P \in \mathcal{O}$ indecomposable projective $\Rightarrow \exists$ simple quotient $L(\lambda) \Rightarrow$

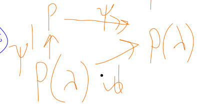
$\exists\, \varphi : P \twoheadrightarrow L(\lambda) \Longrightarrow$



$\exists\, \psi : P \to P(\lambda)$

π_λ essential

$\Rightarrow \psi$ surjective $\Rightarrow$



$\exists\, \psi' : P(\lambda) \to P$ with $\psi \circ \psi' = \mathrm{id}$

$\Rightarrow P(\lambda) \cong P$

Properties of $P(\lambda)$

Theorem

(a) *In $\mathcal{O}$, every indecomposable projective module is isomorphic to a $P(\lambda)$*

(*) (b) *The number of indecomposables isomorphic to $P(\lambda)$ in the decomposition of a projective module $P \in \mathcal{O}$ is equal to $\dim \operatorname{Hom}_{\mathcal{O}}(P, L(\lambda))$.*

$P \in \mathcal{O}$ projective $\overset{(a)}{\Longrightarrow} P = \bigoplus P(\lambda_i)$

$\dim \operatorname{End}(L(\lambda)) = 1 \Longrightarrow \dim \operatorname{Hom}_{\mathcal{O}}(P, L(\lambda)) = \#$ summands isomorphic to $P(\lambda)$

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(b) *The number of indecomposables isomorphic to $P(\lambda)$ in the decomposition of a projective module $P \in \mathcal{O}$ is equal to* $\dim \operatorname{Hom}_{\mathcal{O}}(P, L(\lambda))$.

(c) *For all $M \in \mathcal{O}$,* $\dim \operatorname{Hom}_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)]$.
In particular, $\dim \operatorname{End}_{\mathcal{O}} P(\lambda) = [P(\lambda) : L(\lambda)]$.

Proof

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Definition

The **injective hull** *of* $L(\lambda) \cong L(\lambda)^\vee$ *is* $Q(\lambda) := P(\lambda)^\vee$*.*

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Corollary

The indecomposable injectives in $\mathcal{O}$ *are the modules* $Q(\lambda)$

Standard filtrations of Projectives

Theorem

- *Each projective module in $\mathcal{O}$ has a standard filtration.*

Use construction of (o): $P(\lambda)$ is direct summand of $P = M(\mu) \otimes L$

(Corollary) $\Rightarrow$ P has standard filtration with $M(\lambda)$ occurring just once

and other $M(\nu)$ in filtration have $\nu > \lambda$

($P = P(\lambda) \oplus P'$)

$\Rightarrow$ $P(\lambda)$ inherits same type of filtration

Standard filtrations of Projectives

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- *Each projective module in $\mathcal{O}$ has a standard filtration.*
- *The multiplicity $(P(\lambda) : M(\mu))$ is nonzero only if $\mu \geq \lambda$.*
- $(P(\lambda) : M(\lambda)) = 1$

Proof

Standard filtrations of Projectives

Corollary

Each projective module $P \in \mathcal{O}$ is determined up to isomorphism by its formal character, i.e.,

$$\operatorname{ch} P = \operatorname{ch} P' \text{ implies } P \cong P'.$$

Use (*): P uniquely determines $d_\lambda \in \mathbb{N}$ s.t. $P \cong \bigoplus d_\lambda P(\lambda)$

TP: $\operatorname{ch} P$ determine the d_λ

Exercise 1.16: $\operatorname{ch} P = \sum_\mu c_\mu \operatorname{ch} M(\mu)$, $c_\mu \in \mathbb{Z}$ determined by P

(□) $\Rightarrow c_\mu \in \mathbb{N}$. Let λ be maximal among weight for which $c_\lambda > 0$

$\Rightarrow P(\lambda)$ occurs c_λ times in the decomposition $\Rightarrow d_\lambda = c_\lambda$

Proof

BGG Reciprocity

For all $\lambda \in \mathfrak{h}^*$ we now have three indecomposable modules:

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Theorem (BGG Reciprocity)

For all $\lambda, \mu \in \mathfrak{h}^*$ we have

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)](= [M(\mu)^\vee : L(\lambda)]).$$

$\mathrm{ch}\, M(\mu)^\vee = \mathrm{ch}\, M(\mu)$

Proof

We prove: $[M(\mu)^\vee : L(\lambda)] \overset{(a)}{=} \dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M(\mu)^\vee)$

$$(P(\lambda) : M(\mu)) \overset{(b)}{=} \dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M(\mu)^\vee)$$

(a) (Δ) | (b) $P(\lambda)$ has standard filtration $\Rightarrow$ Use theorem 3.7

Fine points to be investigated

- How de we best describe the socle or radical series of $P(\lambda)$?
- For which λ is $P(\lambda)$ self dual ($P(\lambda) \cong Q(\lambda)$)?
- What is the structure of the algebra $\mathrm{End}_{\mathcal{O}} P(\lambda)$?

$\mathfrak{sl}(2,\mathbb{C})$ example

$\begin{cases} h \cdot v_i = (\lambda - 2i) v_i \\ x \cdot v_i = (\lambda - i + 1) v_{i-1} \\ y \cdot v_i = (i+1) v_{i+1} \end{cases}$

$\mathfrak{h}^* = \mathbb{C}$, $\Lambda = \mathbb{Z}$, $\chi = \chi_\lambda$, $\lambda \in \mathbb{Z} \setminus \{-1\}$

$\Rightarrow$ linked pair: $\lambda, \mu = -\lambda - 2$

set $\lambda \geq 0 \Rightarrow \mu$ antidominant $\Rightarrow L(\mu) = M(\mu)$

λ dominant $\Rightarrow M(\lambda) = P(\lambda)$

Prop 1.4: $\exists$ maximal vect. of weight μ in $M(\lambda)$ (Explicitly $v_{\lambda+1}$)

$\Rightarrow M(\lambda)$ has submodule $\cong L(\mu)$

All weight spaces in $M(\lambda)$ and $L(\mu)$ are 1-dimensional

Composition series $M(\lambda)$: $1 \subset L(\mu) \subset M(\lambda)$ with $L(\lambda) \cong M(\lambda)/L(\mu)$

$M(\lambda)^\vee = P(\lambda)^\vee = Q(\lambda)$: $1 \subset L(\lambda) \subset Q(\lambda)$

$\mathfrak{sl}(2,\mathbb{C})$ example

$P(\mu)$

(□): Standard filtration with $M(\mu) = L(\mu)$ as top quotient

↳ Filtration series: $0 \subset M_1 \subset \ldots \subset M_n = P(\mu)$ with $L(\mu) \cong P(\mu)/M_{n-1}$

$M(\lambda) \cong M_i / M_{i-1} \quad \forall i < n$

BGG-reciprocity: $(P(\mu) : M(\lambda)) = [M(\lambda) : L(\mu)] = 1$

$\Rightarrow 0 \subset M(\lambda) \subset P(\lambda)$ with $L(\mu) \cong P(\mu)/M(\lambda)$

$0 \to M(\lambda) \to P(\mu) \to M(\mu) \to 0$ is nonsplit

$P(\mu) \cong Q(\mu)$

Indecomposable modules for $\mathfrak{sl}(2, \mathbb{C})$

Proposition

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $\{\lambda, \mu\}$ be a linkage class of integral weights with $\lambda \geq 0$, $\mu = -\lambda - 2$ and $\chi = \chi_\lambda$. Every indecomposable module in $\mathcal{O}_\chi$ is isomorphic to one of the following five modules.

$$L(\lambda), \quad L(\mu) = M(\mu), \quad M(\lambda) = P(\lambda), \quad M(\lambda)^\vee = Q(\lambda), \quad P(\mu) = Q(\mu).$$