Category $\mathcal{O}$: Methods

- The functors Hom and Ext
- A duality functor $M \mapsto M^\vee$ on $\mathcal{O}$
- Reflection groups, dominant and antidominant weights
- Tensoring Verma modules with finite dimensional modules
- "Standard" filtrations having Verma modules as subquotients
- Projective objects in $\mathcal{O}$ and BGG reciprocity
- "Contravariant" forms on modules

3. The Reflection group $W_{[\lambda]}$

Known conditions for $L(\mu)$ to be a possible composition factor of $M(\lambda)$:

(1) $\mu \leq \lambda$.

(2) $\mu = w \cdot \lambda$ for some $w \in W$.

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Then we can replace (2) with

(2) $\mu = w \cdot \lambda$ for some $w \in W_{[\lambda]}$.

Remarks

$\{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\ \forall \alpha\}$

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- $\Phi_{[\lambda]} = \{\alpha \in \Phi | s_\alpha \in W_{[\lambda]}\}$

Properties of the reflection group

Theorem

Let $\lambda \in \mathfrak{h}^*$*.*

(a) $\Phi_{[\lambda]}$ *is a root system in its* $\mathbb{R}$*-span* $E(\lambda) \subset E = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$

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(b) $W_{[\lambda]}$ *is the Weyl group of* $\Phi_{[\lambda]}$.
In particular, it is generated by the s_α, *with* $\alpha \in \Phi_{[\lambda]}$.

(a) $s_\alpha \Phi_{[\lambda]} = \Phi_{[\lambda]}$ $\forall \alpha \in \Phi_{[\lambda]}$

Note that $\forall \beta \in \Phi_{[\lambda]}$: $(s_\alpha \beta)^\vee = s_\alpha \beta^\vee$

$\langle \lambda, (s_\alpha \beta)^\vee \rangle = \langle \lambda, s_\alpha \beta^\vee \rangle = \langle s_\alpha \lambda, \beta^\vee \rangle$

$= \langle \lambda, \beta^\vee \rangle - \langle \lambda, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$

Proof

(Anti)dominant weights

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Definition

$\lambda \in \mathfrak{h}^*$ *is* **antidominant** *if* $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ *for all* $\alpha \in \Phi^+$.
$\lambda \in \mathfrak{h}^*$ *is* **dominant** *if* $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{<0}$ *for all* $\alpha \in \Phi^+$.

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Warning: This differs from the usual notion of dominance in Λ.

- Old: Set of dominant weights in Λ is Λ^+.
- New: Set of dominant weights in Λ is $\Lambda^+ - \rho$.

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To avoid confusion we emphasize the antidominance case.

Properties of Antidominant weights

Theorem

Let $\Delta_{[\lambda]}$ be the simple system corresponding to the positive system $\Phi_{[\lambda]} \cap \Phi^+$ in $\Phi_{[\lambda]}$. The following are equivalent.

(1) λ *is antidominant* ($\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ *for all* $\alpha \in \Phi^+$).

(2) $\langle \lambda + \rho, \alpha^\vee \rangle \leq 0$ *for all* $\alpha \in \Delta_{[\lambda]}$.

(3) $\lambda \leq s_\alpha \cdot \lambda$ *for all* $\alpha \in \Delta_{[\lambda]}$.

(4) $\lambda \leq w \cdot \lambda$ *for all* $w \in W_{[\lambda]}$.

(1) $\Rightarrow$ (2) trivial | (2) $\Rightarrow$ (1): $\alpha \in \Phi^+$ and suppose $\langle \lambda+\rho, \alpha^\vee \rangle \in \mathbb{Z}$
$\Rightarrow \alpha \in \Phi_{[\lambda]} \cap \Phi^+ \Rightarrow \alpha$ is $\mathbb{Z}$-linear combination of $\Delta_{[\lambda]}$
$\overset{(2)}{\Rightarrow} \langle \lambda+\rho, \alpha^\vee \rangle \leq 0 \Rightarrow \langle \lambda+\rho, \alpha^\vee \rangle \in \mathbb{Z}^{\leq 0} \Rightarrow (1)$

Proof

(1) λ is antidominant ($\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ for all $\alpha \in \Phi^+$).

(2) $\langle \lambda + \rho, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Delta_{[\lambda]}$.

(3) $\lambda \leq s_\alpha \cdot \lambda$ for all $\alpha \in \Delta_{[\lambda]}$.

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(2) $\Leftrightarrow$ (3)

$$s_\alpha \cdot \lambda = s_\alpha(\lambda + \rho) - \rho = \lambda + \rho - \langle \lambda + \rho, \alpha^\vee \rangle \alpha - \rho$$

$$= \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha \quad (*)$$

(4) $\Rightarrow$ (3) trivial | (3) $\Rightarrow$ (4)

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(3) $\Rightarrow$ (4) Induction on the length in $W_{[\lambda]}$ relative to $\Delta_{[\lambda]}$

$\ell(w) = 0$: $w = 1$ trivial, $\ell(w) > 0$: $w = w' s_\alpha$ in $W_{[\lambda]}$, $\alpha \in \Delta_{[\lambda]}$, $\ell(w') = \ell(w) - 1$

$$\lambda - w\cdot\lambda = \underbrace{(\lambda - w'\cdot\lambda)}_{\leq \text{ IH}} + (w'\cdot\lambda - w\cdot\lambda)$$

$$w'\cdot\lambda - w\cdot\lambda = (w s_\alpha)\cdot\lambda - w\cdot\lambda \overset{(*)}{=} -\underbrace{\langle \lambda+\rho, \alpha^\vee\rangle}_{\leq 0\ (2)} \underbrace{w\alpha}_{0.3(4):\ <0} \leq 0$$

Proof

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Properties of Antidominant weights

Proposition

Any linkage class $W \cdot \lambda$ has at least 1 antidominant weight.

Proof: Suppose $\mu \in W \cdot \lambda$ is minimal relative to standard partial order. But $\exists \alpha \in \Phi^+ : \langle \mu + \rho, \alpha^\vee \rangle \in \mathbb{Z}^{>0}$

$\Rightarrow s_\alpha \cdot \mu \overset{(*)}{=} \mu - \langle \mu + \rho, \alpha^\vee \rangle \alpha < \mu$ ↯ minimality

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Corollary (Exercise)

If $\lambda \in \mathfrak{h}^*$ is antidominant, then $M(\lambda) = L(\lambda)$.

Proof

$\mathfrak{sl}(2,\mathbb{C})$ example

$\mathfrak{g} = \{x, y, h\}$ $\quad \mathfrak{h}^* = \mathbb{C}$ $\quad \langle \lambda, \alpha^\vee \rangle = 2 \cdot \frac{\lambda \cdot \alpha}{\alpha \cdot \alpha}$

$\Phi = \{-2, 2\}$ $\quad \Phi_{[\lambda]} = \{\alpha \in \Phi \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}$

$\Rightarrow \Phi_{[\lambda]} = \Phi$ if $\lambda \in \mathbb{Z} (= \Lambda)$

$\Phi_{[\lambda]} = \emptyset$ if $\lambda \notin \mathbb{Z}$

$W_{[\lambda]} \cdot \lambda = \{\lambda, -\lambda - 2\}$: $\lambda \in \mathbb{Z}^{\geq 0} \Rightarrow -\lambda - 2$ antidominant

$\lambda \in \mathbb{Z}^{<0} \Rightarrow \lambda$ antidominant

$-\rho = 1$

λ antidominant $\Rightarrow \lambda \notin \mathbb{N} \Leftrightarrow M(\lambda) = L(\lambda)$

4. Tensoring Verma modules with fin. dim. modules

Theorem

Let M be a finite dimensional $U(\mathfrak{g})$-module.
For all $\lambda \in \mathfrak{h}^*$, the tensor product $T := M(\lambda) \otimes M$ has a finite filtration with quotients isomorphic to $M(\lambda + \mu)$.
Here μ ranges over the weights of M, each occurring $\dim M_\mu$ times.

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Corollary

T has a submodule isomorphic to $M(\lambda + \mu)$ with μ any maximal weight of M.
T has a quotient isomorphic to $M(\lambda + \nu)$ with ν any minimal weight of M.

Proof

Tensor identity: $(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L) \otimes M \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (L \otimes M)$

Recall: $N \to U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ is exact (if $\dim N < \infty$)

Set $N := L \otimes M$ and order the weight vector basis $v_1, \dots, v_n$ (weights $\nu_1, \dots, \nu_n$) such that $i \leq j$ whenever $\nu_i \leq \nu_j$

$\Rightarrow$ filtration $0 \subset N_n \subset \dots \subset N_1 = N$, $N_k = \langle v_k, \dots, v_n \rangle$

induction on $\dim N \Rightarrow$ RHS has filtration with quotients = Verma modules with weights = weights of N.

Set $L = \mathbb{C}_\lambda \Rightarrow$ LHS $= M(\lambda) \otimes M = T$ and $\dim L \otimes M = \dim M$

weights of $N = L \otimes M$ = weights of M + weight λ of $\mathbb{C}_\lambda$

$\square$

Proof

Exercise

further theory. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and identify $\mathfrak{h}^*$ with $\mathbb{C}$. The reader should be able to describe for each $\lambda \in \mathbf{N}$ the decomposition of $M(0) \otimes L(\lambda)$ relative to central characters χ.

5. Standard filtrations

Definition

$M \in \mathcal{O}$ has a **standard filtration** *(***Verma flag***) if we have*

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for which $M^i := M_i/M_{i-1}$ is isomorphic to a Verma module.

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Denote the multiplicity of a $M(\lambda)$ in a standard filtration of M by

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Not to be confused with the multiplicity of $L(\lambda)$ in a Jordan-Hölder series of M:

$$[M : L(\lambda)]$$

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Suppose $M \in \mathcal{O}$ has a standard filtration.

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(b) Induction on the

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(b) *If* $M = M' \oplus M''$ *in* $\mathcal{O}$*, then* M' *and* M'' *have standard filtrations.*

(c) M *is free as a* $U(\mathfrak{n}^-)$*-module.*

Proof

Universal prop of $M(\lambda)$

(a) By assumption $\exists v_\lambda^+ \in M \Rightarrow \exists \varphi : M(\lambda) \longrightarrow M$

TP: φ is injective | Let i be the smallest index such that

$\varphi(M(\lambda)) \subset M_i \Rightarrow \psi : M(\lambda) \longrightarrow M_i/M_{i-1}$ is nonzero

but $M^i = M_i/M_{i-1} \cong M(\mu) \Rightarrow \lambda \le \mu$ (λ maximal) $\Rightarrow \lambda = \mu$

$\Rightarrow \psi$ is isomorphism $\Rightarrow \varphi$ injective $\Rightarrow M(\lambda) \subset M$

$M(\lambda) \cap M_{i-1} = \ker \psi = 0 \Rightarrow 0 \longrightarrow M_{i-1} \longrightarrow M/M(\lambda) \longrightarrow M/M_i \longrightarrow 0$

Characterization using standard filtrations

Theorem

If $M \in \mathcal{O}$ has a standard filtration, then for all $\lambda \in \mathfrak{h}^*$ we have

$$(M : M(\lambda)) = \dim \operatorname{Hom}_{\mathcal{O}} \left(M, M(\lambda)^{\vee} \right)$$

Proof

Induction on filtration length (3.3(c))

$l=1$: $(M(\mu) : M(\lambda)) = \delta_{\lambda\mu} = \dim \operatorname{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee)$

Induction step: $0 \to N \to M \to M(\mu) \to 0 \Rightarrow$

$0 \to \operatorname{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) \to \operatorname{Hom}_{\mathcal{O}}(M, M(\lambda)^\vee) \to \operatorname{Hom}_{\mathcal{O}}(N, M(\lambda)^\vee) \to \operatorname{Ext}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) \to$

$\dim(\;) = \delta_{\lambda\mu}$; $\dim(\;) = (N : M(\lambda))$ (I.H.) ; $\operatorname{Ext} = 0$ (3.3(d))

$\Rightarrow \dim \operatorname{Hom}(M, M(\lambda)^\vee) = (N : M(\lambda)) + \delta_{\lambda\mu}$

$= (M : M(\lambda))$