Category $\mathcal{O}$: Methods

- The functors Hom and Ext
- A duality functor $M \mapsto M^\wedge$ on $\mathcal{O}$
- Reflection groups, dominant and antidominant weights
- Tensoring Verma modules with finite dimensional modules
- "Standard" filtrations having Verma modules as subquotients
- Projective objects in $\mathcal{O}$ and BGG reciprocity
- "Contravariant" forms on modules

1. Hom and Ext

Hom is left exact:

$$0 \to A \to B \to C \to 0 \text{ exact}$$
$$\implies 0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \text{ exact},$$

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$$\mathrm{Hom} = \mathrm{Ext}^0, \qquad \mathrm{Ext} := \mathrm{Ext}^1$$

Ext

.

We have

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commutes.
The class of split exact sequences

$$[0 \to B \to A \oplus B \to A \to 0]_\sim$$

is the 0 element

Properties

Proposition

Let $\lambda, \mu \in \mathfrak{h}^*$.

(a) *If M is a highest weight module of weight μ, $\lambda \not< \mu$, then $\mathrm{Ext}_{\mathcal{O}}(M(\lambda), M) = 0$. In particular,*

$$\mathrm{Ext}_{\mathcal{O}}(M(\lambda), L(\lambda)) = 0 = \mathrm{Ext}_{\mathcal{O}}(M(\lambda), M(\lambda))$$

$0 \to M \to E \to M(\lambda) \to 0$ in $\mathcal{O}$

Look at preimage of a vector of weight λ in $M(\lambda)$

$\Rightarrow$ it will be maximal vector in E_λ

$\Rightarrow$ It will generate a submodule of E, $\cong M(\lambda)$

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(c) *If* $\mu < \lambda$ *and* $N(\lambda)$ *is the maximal submodule of* $M(\lambda)$, *then*

$$\mathrm{Hom}_{\mathcal{O}}(N(\lambda), L(\mu)) \cong \mathrm{Ext}_{\mathcal{O}}(L(\lambda), L(\mu))$$

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(d) $\mathrm{Ext}_{\mathcal{O}}(L(\lambda), L(\lambda)) = 0$

Proof

(c) $0 \to A \to B \to C \to 0$

$$0 \to \mathrm{Hom}(C,D) \to \mathrm{Hom}(B,D) \to \mathrm{Hom}(A,D)$$
$$\to \mathrm{Ext}(C,D) \to \mathrm{Ext}(B,D) \to \mathrm{Ext}(A,D)$$
$$\to \mathrm{Ext}^2(C,D) \to \cdots$$

$A = N(\lambda)$, $B = M(\lambda)$, $C = L(\lambda)$, $D = L(\mu)$

$\mathrm{Hom}(M(\lambda), L(\mu)) = 0$ ($\mu < \lambda$, $L(\lambda)$ unique simple quotient of $M(\lambda)$)

$\mathrm{Ext}(M(\lambda), L(\mu)) = 0$ (b)

(d) $\mu = \lambda$ $\mathrm{Hom}(N(\lambda), L(\lambda)) = 0$

$\mathrm{Ext}_{\mathcal{O}}$

Note that if we have

$$0 \to A \to B \to C \to 0 \text{ (SES)}$$

with $A, C \in \mathcal{O}$, then B is not necessarily in $\mathcal{O}$, i.e., $\mathrm{Ext}_{\mathcal{O}} \neq \mathrm{Ext}_{U(\mathfrak{g})}$

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Exercise. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, and identify $\lambda \in \mathfrak{h}^*$ with a scalar as usual. Let N be a 2-dimensional $U(\mathfrak{b})$-module defined by letting x act as 0 and h act as $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Show that the induced $U(\mathfrak{g})$-module $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ fits into an exact sequence which fails to split:

$$0 \to M(\lambda) \to M \to M(\lambda) \to 0.$$

(Here M cannot lie in $\mathcal{O}$, thanks to part (a) of the proposition below.)

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Is M a highest weight module?

$\mathfrak{sl}(2,\mathbb{C})$ example

$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) = \{x, y, h\}$, $N = \mathbb{C}^2$ with

$x \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$, $\quad h \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda a + b \\ \lambda b \end{pmatrix}$

$M = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$, recall $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$

TP $\exists$ $0 \to M(\lambda) \xrightarrow{f} M \xrightarrow{g} M(\lambda) \to 0$ which fails to split

· $f: Z \otimes c \longmapsto Z \otimes \begin{pmatrix} c \\ 0 \end{pmatrix}$, $\quad g: Z \otimes \begin{pmatrix} a \\ b \end{pmatrix} \longmapsto Z \otimes b$

suppose $k: M(\lambda) \to M$, $g \circ k = \mathrm{id}_{M(\lambda)}$: $Z \otimes b \longmapsto Z \otimes \begin{pmatrix} \alpha b \\ b \end{pmatrix}$

for some $\alpha \in \mathbb{C}$

$h \otimes b \overset{k}{\longmapsto} h \otimes \begin{pmatrix} \alpha b \\ b \end{pmatrix} = 1 \otimes \begin{pmatrix} \lambda \alpha b + b \\ \lambda b \end{pmatrix}$

$\|$

$1 \otimes \lambda b \overset{k}{\longmapsto} 1 \otimes \begin{pmatrix} \alpha \lambda b \\ \lambda b \end{pmatrix} \Rightarrow$ ~~…~~ $\forall b$

2. Duality in $\mathcal{O}$

For all $M \in \mathrm{Mod}_{U(\mathfrak{g})}$ we have $M^* \in \mathrm{Mod}_{U(\mathfrak{g})}$ with

$$(x \cdot f)(v) := -f(x \cdot v) \quad \forall v \in M,\ f \in M^*,\ x \in \mathfrak{g}.$$

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Recall the transpose map $\tau : \mathfrak{g} \to \mathfrak{g}$ which sends $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$ and fixes $\mathfrak{h}$.
Redefine the action of $\mathfrak{g}$ on M^* as

$$(x \cdot f)(v) := f(\tau(x) \cdot v) \quad \forall v \in M,\ f \in M^*,\ x \in \mathfrak{g}.$$

The dual $M^\wedge$

Let $\mathcal{C} \subset \mathrm{Mod}_{U(\mathfrak{g})}$ be the category of weight modules with finite dimensional weight spaces. ($\mathcal{O} \subset \mathcal{C}$)

For all $\lambda \in \mathfrak{h}^*$ we define $M_\lambda^* := (M_\lambda)^* = (M^*)_\lambda$.

$(M_\lambda)^* = \{ f \in M^* \mid f(v) = 0 \ \forall v \in M_\mu, \mu \neq \lambda \}$

$(M^*)_\lambda = \{ f \in M^* \mid (h \cdot f)(v) = \lambda(h) f(v) \ \forall v \in M \}$

$(M_\lambda)^* \subseteq (M^*)_\lambda$: trivial | $(M^*)_\lambda \subseteq (M_\lambda)^*$:

$f \in (M^*)_\lambda$ then $\mu(h) \neq \lambda(h)$ $\forall v \in M_\mu$:

$\mu(h) f(v) = f(\mu(h) v) = f(h \cdot v) = (h \cdot f)(v) = \lambda(h) f(v) \Rightarrow f(v) = 0$

$(v \in M_\mu)$

$f \in (M^*)_\lambda \Rightarrow f \in (M_\lambda)^*$

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The **dual** of M in $\mathcal{C}$ is

$$M^\wedge := \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda^*$$

Properties of the duality functor

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In particular, $M \in \mathcal{O}_\chi$ implies $M^\wedge \in \mathcal{O}_\chi$.

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(c) *If $M \in \mathcal{O}$ then $\mathrm{ch}\, M = \mathrm{ch}\, M^\wedge$ and define the same element of $K(\mathcal{O})$.*
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(e) *For all $M, N \in \mathcal{O}$, $\operatorname{Ext}_{\mathcal{O}}(M, N) \cong \operatorname{Ext}_{\mathcal{O}}(N^\wedge, M^\wedge)$.*
In particular, $\operatorname{Ext}_{\mathcal{O}}(L(\lambda), L(\mu)) \cong \operatorname{Ext}_{\mathcal{O}}(L(\mu), L(\lambda))$.

Proof

Duals of Highest Weight Modules

Theorem

Let $\lambda, \mu \in \mathfrak{h}^*$.

(a) $L(\lambda)^\wedge \cong L(\lambda)$.

(a) $(L(\lambda)^\vee)^\vee \cong L(\lambda)$ simple $\Rightarrow$ $L(\lambda)^\vee$ simple

$ch\, L(\lambda)^\vee = ch(L(\lambda)) \Rightarrow$ highest of $L(\lambda)^\vee$ is λ

Duals of Highest Weight Modules

Theorem

Let $\lambda, \mu \in \mathfrak{h}^*$.

(a) $L(\lambda)^\wedge \cong L(\lambda)$.

(b) $M(\lambda)^\wedge$ *has* $L(\lambda)$ *as its unique simple submodule. Its other composition factors* $L(\mu)$ *satisfy* $\mu < \lambda$.

.

$$0 \to N(\lambda) \to M(\lambda) \to L(\lambda) \to 0$$

$$\Rightarrow 0 \to L(\lambda)^\vee \to M(\lambda)^\vee \to N(\lambda)^\vee \to 0$$

$$L(\lambda) \cong L(\lambda)^\vee$$

$$\mathrm{ch}\, M(\lambda) = \mathrm{ch}\, M(\lambda)^\vee$$

Duals of Highest Weight Modules

$\mu = \lambda$: $M(\lambda) \to L(\lambda) \hookrightarrow M(\lambda)^\vee$

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Ex 7.3

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(c) $\dim \mathrm{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\wedge) = \delta_{\lambda\mu}$.

Let $f: M(\mu) \to M(\lambda)^\vee$ be a nonzero hom.

$\Rightarrow$ $\mathrm{im}(f)$ = highest weight mod. of weight μ

(b) $\Rightarrow$ $\mathrm{im}(f)$ contains submodule $\cong L(\lambda)^\vee \cong L(\lambda) \Rightarrow \mu \geq \lambda$

(b) also implies $\mu \leq \lambda$ $\Rightarrow \mu = \lambda$

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(d) $\operatorname{Ext}_{\mathcal{O}}(M(\mu), M(\lambda)^\wedge) = 0$

TP $0 \to M(\lambda)^\vee \to M \to M(\mu) \to 0$ splits

If $\mu \not< \lambda$: universal property of Verma modules: splits

$\mu < \lambda$: $0 \to M(\mu)^\vee \to M^\vee \to M(\lambda) \to 0$

proof