Chapter 2

We consider the subcategory of $\mathcal{O}$ consisting of finite dimensional modules.

$$M = \bigoplus_{\lambda \in \Lambda^+} [M : L(\lambda)] L(\lambda).$$

Content:

- Kostant's Weight Multiplicity Theorem
- Weyl's Character Formula
- Weyl's Dimension Formula
- Concrete description of maximal submodule $N(\lambda)$ of $M(\lambda)$, when $\lambda \in \Lambda^+$.
- Related topics about finite dimensional modules in $\mathcal{O}$.

Approach

The BGG approach to Weyl's formulas:

- Think outside of subcategory of finite dimensional modules.
- Generalizes well to other algebraic situations f.x. Kac-Moody algebras.

From 1.16(3):

$$\operatorname{ch} L(\lambda) = \sum_{w \in W} b(\lambda, w) \operatorname{ch} M(w \cdot \lambda).$$

Calculations are done with the two functions:

$$p \in \mathcal{X}, \text{ Kostant function}$$

$$q := \prod_{\alpha > 0} (e(\alpha/2) - e(-\alpha/2)) \in \mathcal{X}.$$

Recall $e(\lambda) \in \mathcal{X}$ is given by $e(\lambda)(\mu) := \delta_{\lambda\mu}$, for all $\lambda, \mu \in \mathfrak{h}^*$.

Approach

From Proposition 1.16 we have $p = \operatorname{ch} M(0)$ and

$$p * e(\lambda) = \operatorname{ch} M(\lambda), \quad (\lambda \in \mathfrak{h}^*).$$

Lemma (2.3(B))

For any $w \in W$, $wq = (-1)^{\ell(w)} q$.

Lemma (2.3(C))

$$q * \operatorname{ch} M(\lambda) = q * p * e(\lambda) = e(\lambda + \rho), \quad (\lambda \in \mathfrak{h}^*).$$

Weyl's Character Formula

Theorem (Weyl)

Let $\lambda \in \Lambda^+$, *so* $\dim L(\lambda) < \infty$. *Then*

$$q * \mathrm{ch}\, L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} e(w \cdot \lambda + \rho).$$

$$q = \sum_{w \in W} (-1)^{\ell(w)} e(w\rho).$$

Proof'ish:

$$q * \mathrm{ch}\, L(\lambda) = \sum_{w \in W} b(\lambda, w) q * \mathrm{ch}\, M(w \cdot \lambda) = \sum_{w \in W} b(\lambda, w) e(w \cdot \lambda + \rho).$$

$\mathrm{ch}\, L(\lambda)$ is W invariant, $s_\alpha e(w \cdot \lambda + \rho) = e(s_\alpha w \cdot \lambda + \rho)$, $s_\alpha q = -q$ and $b(\lambda, 1) = 1$. So induction $\ell(w)$ proves $b(\lambda, w) = (-1)^{\ell(\lambda)}$.

□

Kostant's Weight Multiplicity Theorem

Corollary (Kostant)

If $\lambda \in \Lambda^+$ and $\mu \leq \lambda$, then

$$\dim L(\lambda)_\mu = \sum_{w \in W} (-1)^{\ell(w)} p(\mu - w \cdot \lambda).$$

Proof'ish:

$$q * \operatorname{ch} L(\lambda) = \sum_{\mu \leq \lambda} \sum_{w \in W} (-1)^{\ell(w)} \dim L(\lambda)_\mu e(\mu + w\rho)$$
$$= \sum_{w \in W} (-1)^{\ell(w)} e(w \cdot \lambda + \rho).$$

Now $w\rho = \rho - \sum_\alpha c_\alpha(w)\alpha$. So we look for w' s.t. $\sum_\alpha c_\alpha(w')\alpha = \mu - w \cdot \lambda$. This where $p(\mu - w \cdot \lambda)$ enters.

□

Weyl's dimension Formula

Theorem (Weyl)

$$\dim L(\lambda) = \frac{\prod_{\alpha>0} \langle \lambda + \rho, \alpha^v \rangle}{\prod_{\alpha>0} \langle \rho, \alpha^v \rangle}, \quad (\lambda \in \Lambda^+).$$

Proof'ish: Let $\mathcal{Y}$ be the subring of $\mathcal{X}$ generated by $e(\mu)$. Consider

$$v : \mathcal{Y} \to \mathbb{Z}, \quad f \mapsto \sum_{\mu} f(\mu), \quad \operatorname{ch} L(\lambda) \mapsto \dim L(\lambda).$$

Construct for each $\alpha > 0$ a derivation

$$\partial_\alpha : \mathcal{Y} \to \mathcal{X}_0, \quad \partial_\alpha e(\mu) = \langle \mu, \alpha^v \rangle e(\mu).$$

Set $\partial := \prod_{\alpha>0} \partial_\alpha$ and prove $v(\partial(q * \operatorname{ch} L(\lambda))) = v(\partial q) \dim L(\lambda)$.

$$\dim L(\lambda) = \frac{v(\partial \sum_{w \in W} (-1)^{\ell(w)} e(w \cdot \lambda + \rho)))}{v(\partial q)} = \frac{\prod_{\alpha>0} \langle \lambda + \rho, \alpha^v \rangle}{\prod_{\alpha>0} \langle \rho, \alpha^v \rangle}.$$

□

Maximal submodules

Theorem

If $\lambda \in \Lambda^+$, the maximal submodule $N(\lambda)$ of $M(\lambda)$ is the sum of the submodules $M(s_\alpha \cdot \lambda)$, for $\alpha \in \Delta$.

Proof'ish: Enumerate the simple roots $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ and write $n_i := \langle \lambda, \alpha_i^\vee \rangle$. We know that $M(\lambda) = U(\mathfrak{g})/I$, where

$$I = \text{ Left ideal generated by } \mathfrak{n} \text{ and } h - \lambda(h),\ (h \in \mathfrak{h}).$$

Set then $M = U(\mathfrak{g})/J$, where

$$J = \text{ Left ideal generated by } I \text{ and } y_{\alpha_i}^{n_i+1},\ 1 \leq i \leq \ell.$$

- Note, J is the left ideal generated by the sum of $M(s_\alpha \cdot \lambda)$.
- Show that M is finite dimensional. This implies $M \cong L(\lambda)$.
- From Exercise 1.6, M is finite dimensional if y_{α_i} acts locally nilpotently on M for all $i \in \{1, \ldots, \ell\}$. □