Review

ρ: half sum of weight

Prop 1.4'

if $\alpha \in \Delta$, $\lambda \in \mathfrak{h}^{\vee}$, $\langle \lambda+\rho, \alpha^{\vee} \rangle \in \mathbb{Z}^{>0}$

There exists

$$M(s_\alpha \cdot \lambda) \longrightarrow N(\lambda) \subset M(\lambda)$$

$$w \cdot \lambda = w(\lambda+\rho) - \rho$$

Theorem Harish-Chandra

a) $Z(\mathfrak{g}) \xrightarrow[\sim]{\Psi} S(\mathfrak{h})^W$

b) $\chi_\mu = \chi_\lambda$ if and only if $\mu = w \cdot \lambda$ for a $w \in W$

c) Every central character $\chi : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ is a $\chi = \chi_\lambda$, $\lambda \in \mathfrak{h}^{\vee}$

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Theorem

Category $\mathcal{O}$ is artinian.

1) $M \in \mathcal{O}$ is artinian

2) $\dim \mathrm{Hom}_{\mathcal{O}}(M, N) < \infty$ $M, N \in \mathcal{O}$

Idea of Proof - Prove it for Verma module.

take $V = \sum_{w \in W} M(\lambda)_{w \cdot \lambda}$

If $N' \subset N \subset M(\lambda)$. are submodules

1) $\dim V < \infty$ (weight module)

N/N' has a maximal vector of weight $\mu \leq \lambda$ and $Z(\mathfrak{g})$ acts by χ_λ on N/N' So $\chi_\mu = \chi_\lambda$

by Harish-Chandra Theorem, b) $\mu = w \cdot \lambda$

So $N \cap V \neq \emptyset$

$N \cap V' \neq \emptyset$ implies

so $\dim V \cap N > \dim V \cap N'$

These dimension are finite So a descending chain of submodule will end.

Corollary

Each $M \in \mathcal{O}$ possesses a composition series of finite lenght with simple quotients isomorphic to some $L(\lambda)$.

Definition

The Grothendieck group of $\mathcal{O}$ is $\mathrm{AB}\langle [M] \mid M \in \mathcal{O} \rangle$ / $[B] = [A] + [C]$ if $0 \to A \to B \to C \to 0$ exact

Composition series

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset M$$

with M_i / M_{i-1} simple

By Jordan Hölder theorem, we can speak of the composition series, because all simple quotient appears in any composition series.

We write $[M(\lambda) : L(\mu)]$ for the number of simple quotient isomorphic to $L(\mu)$ in $M(\lambda)$.

Socle M, $\mathrm{Soc}\,M$: sum of simple submodule of M

radical: $\mathrm{Rad}\,M$: intersection of maximal submodule

head : $\mathrm{Hd}\,M$: quotient $M/\mathrm{Rad}\,M$.

1/2

Definition χ central character

$$M^{\chi} := \{ v \in M \mid (z - \chi(z))^{n} \cdot v = 0 \text{ for } n \gg 0,\ n = n(z),\ z \in Z(\mathfrak{g}) \}.$$

$\mathcal{O}_{\chi} \subset \mathcal{O}$. subcategory of objects $M = M^{\chi}$

Proposition

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^{*} / \text{mid } W} \mathcal{O}_{\chi_{\lambda}}$$

$\to$ orbital dot action

Indecomposable module lies in one $\mathcal{O}_{\chi_\lambda}$

comment

χ is a central character that may be expressed as χ_λ one for each W. orbit.

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Definition

If M_1 and M_2 are such that there is a M with

$$0 \to M_1 \to M \to M_2 \to 0$$

$$0 \to M_2 \to M \to M_1 \to 0$$

short exact non-split sequences then M_1 and M_2 are in the same block.

Prop If $\lambda \in \Lambda$ (integral) then $\mathcal{O}_{\chi_\lambda}$ is a block

M arbitrary

M belongs to a block if all its composition factors belong to it

$\exists$ block $\subset \mathcal{O}_{\chi_\lambda}$

*
↳ Try to see it

$\boxed{\text{That each indecomposable module belongs to a block is \underline{not} obvious and follows from artinian condition.}}$

Idea: We use Prop 1.4 to get that all $L(w \cdot \lambda)$ are in the same block.

$$L(w \cdot \lambda) = M(w \cdot \lambda) / N(w \cdot \lambda) \hookrightarrow N(\lambda) / \operatorname{rm} W(w \cdot \lambda)$$

The block $\mathcal{O}_{\chi_0}$ is called the principal block.

The proposition fails for non-integral weights

$\mathcal{O}$ is integral in $\mathfrak{h}^*$

$\mathfrak{sl}_2$ Verma module

when λ was positive but not integral, we still had .

$M(\lambda)$ simple..

but λ and $-\lambda-\alpha$ were ~~W~~- linked

so $L(\lambda)$ and $L(-\lambda-\alpha) \in \mathcal{O}_{\chi_\lambda}$

but they are different. so $\mathcal{O}_{\chi_\lambda}$ not a block

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Definition

Let $e(\lambda)$ be a symbol associated to $e(\lambda) \in \mathbb{Z}\Lambda$. The formal character of M lying in $\underline{\mathbb{Z}\Lambda}$ is

$$\operatorname{ch} M := \sum_{x \in \Lambda} \dim M_x \, e(x)$$

$$\operatorname{ch}(M \otimes N) = \operatorname{ch} M \cdot \operatorname{ch} N$$

M finite-dimensional

If $M \in \mathcal{O}$, $\operatorname{ch} M$ is well defined as a finite sum in $\mathbb{Z}\Lambda$

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For the possibly infinite-dimensional cases we need:

ch M $\longrightarrow$ $\mathbb{Z}^+$ valued function on $\mathfrak{h}^*$

$e(\lambda) \longrightarrow$ characteristic function

$$e(\mu) = \begin{cases} 1 & \mu = \lambda \\ 0 & \mu \neq \lambda \end{cases}$$

Multiplication $\longrightarrow$ convolution

Definition The group of functions $f : \mathfrak{h}^* \longrightarrow \mathbb{Z}$ with support lies in a finite union of $\lambda - \Gamma$ (OS) is denoted $\mathcal{X}$

all $e(\lambda) \in \mathcal{X}$ and e_0 is identity under convolution.

$\mathcal{X}_0 \subset \mathcal{X}$ subgroup generated by $\operatorname{ch} M \mid M \in \mathcal{O}$.

$M \in \mathcal{O}$

Work as functions of $\mathfrak{h}^*$, not simply as element of ring

$e(\lambda)$ a function.

$$(f * g)(\lambda) = \sum_{\mu + \nu} f(\mu) g(\nu)$$

$\mathcal{X}$: group of function

$f : \mathfrak{h}^* \longrightarrow \mathbb{Z}$,

$\rightarrow$ The set of weight is contained in a finitely many sets of the form $\lambda - \Gamma$, Γ semigroup in Λ_r generated by Φ^+

$$e_0 * f(\lambda) = \sum_{\mu + \nu = \lambda} e_0(\mu) f(\nu) = f(\lambda)$$

Proposition ch M respects:

a) $0 \to M' \to M \to M'' \to 0$ short exact sequence then

$$\operatorname{ch} M = \operatorname{ch} M' + \operatorname{ch} M''$$

b) $\mathcal{X}_0 \xrightarrow{\sim} K(\mathcal{O})$

$$\operatorname{ch} M \mapsto [M]$$

c) if $M \in \mathcal{O}$, L fin.-dim.

$$\operatorname{ch}(L \otimes M) = \operatorname{ch} L * \operatorname{ch} M$$

Comments

a) If we have such a short exact sequence, we will have

$$\dim M_\mu = \dim M'_\mu + \dim M''_\mu$$

($L \otimes M$ is in $\mathcal{O}$)

$$(L \otimes M)_\lambda = \sum_{\mu+\nu} L_\mu \otimes M_\nu$$

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M(λ) Verma Module

Definition The Kostantfunction $\rho \in \mathcal{X}$ is

$$\rho(\gamma) = \#\ \text{families}\ (c_\alpha)_{\alpha>0},\ c_\alpha \in \mathbb{Z}^+$$

$$\gamma = -\sum_{\alpha>0} c_\alpha \alpha$$

Proposition for $\lambda \in \tilde{\mathfrak{h}}^+$

$$\mathrm{ch}\, M(\lambda) = \rho * e(\lambda)$$

$$\mathrm{ch}\,(0) = \rho.$$

More info are needed on
$\mathrm{ch}\, L(\lambda)$.

Write
$$\mathrm{ch}\, M(\lambda) = \mathrm{ch}\, L(\lambda) + \sum_{\substack{\mu \leq \lambda \\ \mu = w \cdot \lambda}} [M(\lambda) : L(\mu)]\, \mathrm{ch}\, L(\mu)$$

Invert it (Triangular)

$$\mathrm{ch}\, L(\lambda) = \sum_{w \cdot \lambda \leq \lambda} b(\lambda, w)\, \mathrm{ch}\, M(w \cdot \lambda)$$

$b(\lambda, w) \in \mathbb{Z}$

$\mathfrak{sl}_2$

$\lambda < 0$, we know $M(\lambda) = L(\lambda)$

$\lambda \geq 0$ we know we have one linkage $M(-\lambda-2)$ so

$\mathrm{ch}\, L(\lambda) = \mathrm{ch}\, M(\lambda) - \mathrm{ch}(M(-\lambda-2))$?

for $\lambda \in \Lambda^+$, chapter 2

$\lambda < 0$. $M(\lambda)$ is simple

if $\lambda \notin \mathbb{Z}^+$, then $M(\lambda)$ is simple also $M(\lambda) = L(\lambda)$

$\lambda > 0$ $-\lambda-2$ is linked by the W action to λ

if $\lambda \in \mathbb{Z}^+$ the their which in in chapter

it is finite-dimensional