1.5 $\mathfrak{sl}_2$ 02-02-2021

Standard basis $\underset{H}{h}, \underset{n}{x}, \underset{n^-}{y}$

$[h, x] = 2x$ $\quad [h, y] = -2y$ $\quad [x, y] = h.$

$H^* \longrightarrow \mathbb{C}$ (it is 1-dimensional)

$\Lambda = \mathbb{Z}$ $\quad \Lambda_r = 2\mathbb{Z}.$

Consider $M(\lambda)$: Verma module of weight λ.

1) Weight of $M(\lambda)$ are $\lambda, \lambda-2, \ldots$

v_+ ~~the~~ maximal vector of weight λ, $h \cdot v_+ = \lambda v^+$, $x \cdot v_+ = 0$

$h \cdot (y \cdot v_+) = y \cdot h v_+ - 2 y v_+ = \underline{(\lambda - 2)} y v^+$

2) A basis is given by $\{ v_i \mid i \geq 0 \}$, $v_{-1} := 0$

$h \cdot v_i = (\lambda - 2i) v_i$

$x \cdot v_i = (\lambda - i + 1) v_{i-1}$

$y v_i = (i+1) v_{i+1}$

Put $v_i = y^i v^+ / (i)!$

$y v_i = y \cdot y^i v^+ / i! = y^{i+1} v^+ / i! = (i+1) v_{i+1}$

$h v_i = h \cdot y^i v^+ / (i)! = \underset{[h, y] = -2y}{\circ\circ\circ} = (\lambda - 2i) v_i$

$x \cdot v_1 = x \cdot y v^+ / 1! = (y \cancel{x} + h) v^+ = h v^+ = \lambda v^+ = (\lambda - 1 + 1) v_0$

3) Dim $L(\lambda)<\infty$ if and only if $\lambda\in\mathbb{Z}_+$. And then $N(\lambda)\simeq L(-\lambda-2)$

$\Leftarrow$) Use prop 1.4 to get $\underbrace{M(-\lambda-2)}_{L(-\lambda-2)} \longrightarrow N(\lambda)$.

and thus

$L(\lambda)\simeq M(\lambda)/M(-\lambda-2)$ finite dim

$\to \langle\lambda,\alpha^\vee\rangle\in\mathbb{Z}^+$

if $\lambda\in\mathbb{Z}^+$

$\Rightarrow$) use 4) to have $M(\lambda)\simeq L(\lambda) \Rightarrow \dim L(\lambda)=\infty$

if $\lambda\notin\mathbb{Z}^+$.

4) $M(\lambda)$ is simple if and only if $\lambda\notin\mathbb{Z}^+$

$\Rightarrow$) contrapositive

$\lambda\in\mathbb{Z}^+$ then prop 1.4 tells us that $M(\lambda)$ has a maximal ~~e~~submodule.

$\Leftarrow$ If $\lambda\notin\mathbb{Z}^+$ then any element of the basis generates $M(\lambda)$

Use the basis found earlier.

Exercice For $\mathfrak{g}=\mathfrak{sl}_2$ show that $M(\lambda)\otimes M(\mu)\notin\mathcal{O}$.

What fails is finitely-generated (O1).

Alternatively:

Assume $M\in\mathcal{O}$ has finite Jordan-Hölder lenght (4.11)

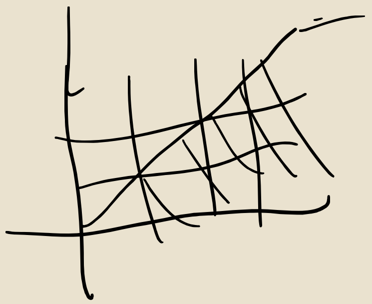
with quotient $\simeq L(\lambda)$, a finite number of them.

Consider the weight space of the tensor product

$M=M(\lambda)\otimes M(\mu)$ – $M_\eta = \bigoplus_{\mu'+\lambda'=\eta} M(\lambda)_{\lambda'}\otimes M(\mu)_{\mu'}$

$\lambda-j\oplus\mu-j$ $\quad\lambda-\ \ \lambda-2\ \ \lambda-4 - -\quad \mu-\ \ \mu-2\ \ \mu-4 - .$

In particular this grows bigger than the Jordan-Hölder lenght. So you would get a contradiction.



Rightest one grows big! by 1 in each step.

up to a careful

1.6

Let, for each $\alpha \in \Phi^+$ $\mathfrak{s}_\alpha$ be the copy of $\mathfrak{sl}_2$ spanned by h_α x_α y_α.

Theorem The simple module $L(\lambda) \in \mathcal{O}$ is finite-dimensional if and only if $\lambda \in \Lambda^+$, or equivalently if and only if $\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu}$ $\forall \mu \in \mathfrak{h}^*$ and $\forall w \in W$.

A

B

Proof

$A \Rightarrow$ ok, $\lambda(h_i) \in \mathbb{Z}^+ \Rightarrow \lambda \in \Lambda^+$

$A \Leftarrow$ Difficult. use coro 1.4 then look at $\underline{\underline{\mathfrak{s}_i}}$ → Not $\mathfrak{g}$! – submodule of $L(\lambda)$ for finite dim.

$B \Rightarrow$ arrives in $A \Leftarrow$.

$B \Leftarrow$ immediate because then $L(\lambda)$ has finitely many weights, so fin-dim.

Properties $\lambda \in \Lambda^+$

λ dominant + integral so weights of $L(\lambda)$ are W-conjugate to $\mu \leq \lambda$.

a) Let $\mu = w \cdot \lambda$ with $w \in W$. Not both $\mu - \alpha$ and $\mu + \alpha$ can occur as weight for any $\alpha \in \Phi$

b) If μ and $\mu + k\alpha$ are weights of $L(\lambda)$, so are the $\mu + i\alpha$ $i = 0, \dots, k$

c) the dual space $L(\lambda)^*$ with standard action $(x \cdot f)(v) = -f(x \cdot v)$ $x \in \mathfrak{g}$, $v \in L(\lambda)$ $f \in L(\lambda)^*$ is isomorphic to $L(-w_0\lambda)$

$w_0 \in W$: longest element.

1.7.

Contrary to finite-dimensional case, we need more than just a Casimir, we need to investigate the whole center of $\mathcal{U}(\mathfrak{g})$, $Z(\mathfrak{g})$

Suppose M is a highest-weight module of maximal vector v^+ of weight λ.
for $z \in Z(\mathfrak{g})$, $h \in \mathfrak{h}$

$$h \cdot (z \cdot v^+) = z \cdot (h v^+) = z \cdot \lambda(h) v^+ = \lambda(h) z v^+$$

So $z \cdot v^+ = \chi_\lambda(z) v^+$ for $\chi_\lambda(z) \in \mathbb{C}$ because $\dim M_\lambda = 1$, M HWM

Definition For $\lambda \in \mathfrak{h}^*$ the central character of λ, χ_λ is an algebra morphism

$$\chi_\lambda : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$$
$$z \longmapsto \chi_\lambda(z)$$

any $\chi : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ is called central character

Remark Ker $Z(\mathfrak{g})$ is a maximal ideal of $Z(\mathfrak{g})$

$$\left\{ \chi \text{ central character} \right\} \longleftrightarrow \left\{ \text{maximal ideal of } Z(\mathfrak{g}) \right\}$$

Let $z \in Z(\mathfrak{g})$ be written by PBW monomials:

Any of those with $\mathfrak{n}^+$ kill v^+ $\mathfrak{n}^+$

those of $\mathfrak{h}$ multiply v^+ by scalars $\mathfrak{h}$

those of $\mathfrak{n}^-$ lower the weight $\mathfrak{n}^-$

$z \cdot v^+$ depends only on monomials with components only in $\mathfrak{h}$.

$pr : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$

Projection to $\mathcal{U}(\mathfrak{h})$. It keeps PBW monomials with only $\mathfrak{h}$ factors

$$\chi_\lambda(z) = \lambda(pr(z)) \qquad \forall z \in Z(\mathfrak{g})$$

↳ λ extended to $\lambda : \mathcal{U}(\mathfrak{h}) \to \mathbb{C}$ algebra morphism

Definition

$$\xi : Z(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$$

$$\mathrm{Res}^{\mathcal{U}(\mathfrak{g})}_{Z(\mathfrak{g})}(pr)$$

is called the Harish-Chandra homomorphism.

⚠ other people call Humphreys's Twisted H-C, the H-C morphism.

Prop It is an algebra morphism, $\lambda : \mathcal{U}(\mathfrak{h}) \to \mathbb{C}$

Humphreys says it follows from $\bigcap_{\lambda \in \mathfrak{h}^*} \mathrm{Ker}\,\lambda = 0$

Also follows from looking at $\mathcal{U}(\mathfrak{g})_0$ centralizer of $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$

- Includes $Z(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{h})$.
- $C := \mathcal{U}(\mathfrak{g})\mathfrak{n} \cap \mathcal{U}(\mathfrak{g})_0 = \mathfrak{n}^-\mathcal{U}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g}_0)$
  - Two-sided ideal of $\mathcal{U}(\mathfrak{g})$
  - Complement to $\mathcal{U}(\mathfrak{h})$
- So $pr\ \mathcal{U}(\mathfrak{g})_0 \to \mathcal{U}(\mathfrak{h})$ goes to $\xi : Z(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$

  because then z projected in this $\mathcal{U}(\mathfrak{g})_0$ will have a part of $Z(\mathfrak{g})$ and no part in $\mathcal{U}(\mathfrak{g})\mathfrak{n} \cap \mathfrak{n}^-\mathcal{U}(\mathfrak{g})$ when acting on highest weight vector (killed by $\mathfrak{n}$)

  So a true morphism

1.8

We want to understand exactly when $\chi_\lambda = \chi_\mu$.

We have a clue from Prop 1.4 which says that if $n = \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^+$ then $\chi_\lambda = \chi_{\lambda-(n+1)\alpha}$

In this case, take ρ: half-sum of positive roots,
Then $s_\alpha(\lambda+\rho) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha + \rho - \alpha = \mu + \rho$.

$$s_\alpha(\rho) = \rho - \alpha$$

This leads to a new action and the definition thereafter of a twisted Harish-Chandra module

Definition The <u>dot action</u> is

$$w \cdot \lambda := w(\lambda+\rho) - \rho \qquad \text{for } w \in W, \lambda \in \mathfrak{h}^*$$

We say λ and μ are <u>linked</u> if there is a w such that $\mu := w \cdot \lambda$. $w \in W$

linkage is an equivalence relation; the orbit $\{w \cdot \lambda / w \in W\}$ is the <u>linkage class</u> of λ

Regular weight or <u>dot-regular weight</u>, are those such that

$$|W \cdot \lambda| = |W|, \quad \text{so} \quad \langle \lambda+\rho, \alpha^\vee \rangle \neq 0 \quad \forall \alpha \in \Phi$$

Exercice (non-additivity of the dot action)

a) $w \cdot (\lambda + \mu) = w(\lambda + \mu + \rho) - \rho$

$$= w\lambda + w\mu + w\rho - \rho$$

$$= w \cdot \lambda + \underset{Z}{w\mu} = w\lambda + w \cdot \mu$$

b) $w\cdot\lambda - w\cdot\mu = \begin{matrix} w(\lambda+\rho)-\rho \\ - w(\mu+\rho)-\rho \end{matrix} = w(\lambda-\mu)$

In this new vocabulary, we rewrite Prop 1.4

Prop 1.4' If $\alpha\in\Delta$, $\lambda\in\mathfrak{h}^*$ and $\langle\lambda+\rho,\alpha^\vee\rangle=0$ then $M(s_\alpha\cdot\lambda)=M(\lambda)$. If $\langle\lambda+\rho,\alpha^\vee\rangle\in\mathbb{Z}^{>0}$ then

$$M(s_\alpha\cdot\lambda)\longrightarrow N(\lambda)\subset M(\lambda).$$

Prop If $\lambda\in\Lambda$ $\boxed{(\Lambda \text{ integral}),}$ and $\mu=w\cdot\lambda$ then $\chi_\lambda=\chi_\mu$

Proof~ $n=\langle\lambda,\alpha^\vee\rangle\in\mathbb{Z}.$ $\longleftarrow$ ~~Regularity~~ Integral

if n is positive Prop 1.4

if n is negative & $n=-1$ would imply that $s_\alpha\cdot\lambda=\lambda=\mu$ so $\chi_\lambda=\chi_\mu$

if $n\leq-2$, then take $\mu=s_\alpha\cdot\lambda$ $(\lambda=s_\alpha\cdot\mu)$

$\langle\mu,\alpha^\vee\rangle=-n-2$ so for this we have

Proposition 1.4'

This shows it for simple roots and iterated in general.

Prop + Proposition holds for all $\lambda\in\mathfrak{h}^*$, no ~~regularity~~ integrality needed.

Proof (skipped): It uses density arguments

1.9

We study here the (Twisted) Harish-Chandra morphism

<> often

„ $\mathcal{U}(\mathfrak{h}) \simeq P(\mathfrak{h}^*)$

Definition Denote $S(\mathfrak{h})$ the algebra of polynomial function in l / rank of $\mathfrak{g}$ variables.

Definition The map $\psi : Z(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ defined by

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r, "\overset{H-C\,\chi}{\xi}"] & \mathcal{U}(\mathfrak{h}) \simeq S(\mathfrak{h}) \arrow[r, "p(\lambda) \longmapsto p(\lambda+\rho)"] & S(\mathfrak{h}) \end{tikzcd} \quad - P(\mathfrak{h}^*)$$

is called the (Twisted) Harish-Chandra homomorphism

$$\chi_\lambda(z) = (\lambda+\rho)(\psi(z)) \qquad \forall z \in Z(\mathfrak{g})$$

Theorem

a) $\lambda, \mu \in \mathfrak{h}^*$ linked then $\chi_\lambda = \chi_\mu$

b) $\operatorname{Im}\psi \subset S(\mathfrak{h})^W$ with the W action here the dot action

We can see that ψ is invariant/constant on the W-orbits

Ex

1.10

Theorem (Harish-Chandra) Let $\Psi : Z(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ be the twisted Harish-Chandra homomorphism

~> a) Ψ is an isomorphism $Z(\mathfrak{g}) \xrightarrow{\simeq} S(\mathfrak{h})^W$

b) $\forall \lambda, \mu \in \mathfrak{h}^*$, $\chi_\lambda = \chi_\mu$ if and only if $\mu = w \cdot \lambda$ for a $w \in W$

c) Every morphism $\chi : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ has the form χ_λ for a λ

Proof a) The idea is to compare with a map

$$\Theta : P(\mathfrak{g}) \longrightarrow P(\mathfrak{h})$$

(polynomial funct on $\mathfrak{g}$, vector space)

is a algebra morphism.

↪ Look now at the adjoint group

$$G = \langle \exp(\mathrm{ad}\, x) \mid x \in \mathfrak{g} \text{ nilpotent} \rangle \subset \mathrm{Aut}\, \mathfrak{g}$$

↪ Lie group, it acts on $P(\mathfrak{g})$, and will have an algebra of fixed points $P(\mathfrak{g})^G$

Now Chevalley Restriction theorem

$$\Theta[P(\mathfrak{g})^G] \simeq P(\mathfrak{h})^W$$

Claim: if you look at graded version of Θ and ξ, you will be able to conclude that Ψ is indeed isomorphism

b) <= known. Now we want $\chi_\mu = \chi_\lambda \Rightarrow \mu = w \cdot \lambda$. Prove contraposition. Suppose μ is not linked to λ. Then take a function $f : \begin{cases} 1 \text{ on } W(\lambda+\rho) \\ 0 \text{ on } W(\mu+\rho) \end{cases}$

$$f' = \frac{1}{|W|} \sum_{w \in W} w.f \quad \text{is } W\text{-invariant.}$$

Take the inv of a). So we get a $z \in Z(g)$ such that $\Psi(z) = f'$

$$\chi_\lambda(z) = (\lambda+\rho)(\Psi(z)) = f'(\lambda) = 1$$

$$\chi_\mu(z) = (\mu+\rho)(\Psi(z)) = f'(\mu) = 0$$

They are not equal.

c)

$$\begin{tikzcd} \chi \arrow[r, Rightarrow, "\Psi"'] & \varphi : S(H)^W \longrightarrow \mathbb{C} \arrow[r, Rightarrow, "\text{algebraic argument}"] & \hat{\varphi} : S(h) \to \mathbb{C} \end{tikzcd}$$

Going up then evaluate

$$\chi(z) = (\lambda+\rho)\ \Psi(z) = \chi_\lambda(z)$$

1.11

Theorem $\mathcal{O}$ is artinian

Proof For Verma Modules