Alexis E-R

Basics of category $\mathcal{O}$

Preliminaries 1.1

We work in a subcategory of $\mathrm{Mod}\,\mathcal{U}(\mathfrak{g})$, the category of (left)-modules of $\mathcal{U}(\mathfrak{g})$.

In $\mathcal{U}(\mathfrak{g})$. We get from

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

* Not only fin-dim, despite the notation

a PBW basis for $\mathcal{U}(\mathfrak{g})$

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-)\,\mathcal{U}(\mathfrak{h})\,\mathcal{U}(\mathfrak{n}^+)$$

It will contains all finite-dimensional modules, and two other families.

Definition The Bernstein–Gelfand–Gelfand category $\mathcal{O}$ is the full subcategory of $\mathrm{Mod}\,\mathcal{U}(\mathfrak{g})$, where for $M \in \mathrm{Obj}\,\mathcal{O}$

O1) M is a finitely generated $\mathcal{U}(\mathfrak{g})$-module.

O2) M is $\mathfrak{h}$-semisimple: $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, $M_\lambda := \{v \in M \mid h\cdot v = \lambda(h)v,\ h \in \mathfrak{h}\}$
(it is a weight module)

O3) M is locally $\mathfrak{n}$-finite: for all $v \in M$ $\mathcal{U}(\mathfrak{n})\cdot v$ is finite-dimensional.

Finite-dimensional modules are in $\mathcal{O}$ because by Weyl's complete reducibility Theorem, they are a direct sum of weight spaces, and $\mathfrak{n}^-$ and $\mathfrak{n}^+$ act nilpotently, so O1) – O3) are satisfied.

Two more properties $\mathcal{O}$ satisfies directly two more properties

$\mathcal{O}4$) All weight spaces are finite-dimensional

$\mathcal{O}5$) The set of $\Pi(M) = \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$ is contained in the union of finitely many $\lambda - \Gamma$, with $\Gamma \subset \Lambda_r$ the semigroup generated by Φ^+. (Λ_r lattice of Φ)

We get them by contemplating the PBW basis.

Proposition Category $\mathcal{O}$ satisfies

a) $\mathcal{O}$ is a noetherian category: $M \in \mathcal{O}$ is noetherian (Ascending Chain condition)

b) $\mathcal{O}$ is closed under submodule, quotient and finite direct sum

⚠ Not tensor product

c) $\mathcal{O}$ is an abelian category

d) If L is finite-dimensional $L \otimes M \in \mathcal{O}$ ($\mathcal{O} \xrightarrow{L\otimes -} \mathcal{O}$ exact endofunctor)

e) M is $Z(\mathfrak{g})$-finite sp $\{z \cdot v \mid z \in Z(\mathfrak{g})\}$ is finite-dimensional $\forall v \in M$.

f) M is finitely generated as a $\mathcal{U}(\mathfrak{n}^-)$ module

1.2

Vocabulary $M \in \mathcal{U}(\mathfrak{g})$. M_λ weight space of $\lambda \in \mathfrak{h}^*$.

- $v^+ \in M$ a maximal vector of weight $\lambda \in \mathfrak{h}^*$ if $v^+ \in M_\lambda$ and $\mathfrak{n}^+ \cdot v^+ = 0$
- M is a highest weight module if there exists $v^+ \in M$ maximal vector, and $M = \mathcal{U}(\mathfrak{g}) \cdot v^+$

Prop $M \in \mathcal{O}$. $\mathcal{H}$. highest weight module

a) There exist a maximal vector in M.

b) $\mathcal{H} \in \mathcal{O}$.

Highest weight modules are important because of the following

Prop. Let $M \in \mathcal{O}$. M admits a finite filtration

$$0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \cdots \hookrightarrow M_n = M$$

where M_i / M_{i-1} is a highest weight module (non-zero)

This comes by induction and the closeness of $\mathcal{O}$ under quotient.

Theorem (Properties of highest weight modules)

Let M be a highest weight module of weight $\lambda \in \mathfrak{h}^*$ with maximal vector v^+
fix positive roots $\alpha_1 \cdots \alpha_\ell$ and denote $\alpha_i \longleftrightarrow y_i \in \mathfrak{g}_{-\alpha_i}$

a) M is a semisimple $\mathfrak{h}$-module spanned by $y_1^{i_1} \cdots y_\ell^{i_\ell} v^+$, $i_j \in \mathbb{Z}^+$ of weight $\lambda - \sum_{j=1}^{\ell} i_j \alpha_j$

(?) b) All weights μ of M satisfy $\mu \leq \lambda$: $\mu = \lambda -$ (sum of positive roots)

c) For all weight μ of M. $\dim M_\mu < \infty$, $\dim M_\lambda = 1$; M is a weight module
locally $\mathfrak{n}$-finite, so $M \in \mathcal{O}$.

d) Each non-zero quotient of M is again a highest weight module of weight λ.

e) Each submodule of M is a weight module. If w^+ is maximal
of weight $\mu < \lambda$, then $\langle w^+ \rangle \subsetneq M$; if M is simple, all
maximal vectors are multiple of v^+

f) M is indecomposable: it has a unique maximal submodule and a unique
quotient.

g) All simple highest weight module of weight λ are isomorphic
If M is, $\dim(\mathrm{End}_{\mathcal{O}} M) = 1$

1.3

We want to use the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ to induce a family of modules in $\mathcal{O}$. We use induction

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} : \mathrm{Mod}(\mathcal{U}(\mathfrak{b})) \longrightarrow \mathrm{Mod}(\mathcal{U}(\mathfrak{g}))$$
$$M \longmapsto \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} M$$

Definition Let $\mathbb{C}_\lambda$ be the 1-dimensional $\mathfrak{b}$-module linked to the weight $\lambda \in \mathfrak{h}^*$. The Verma module linked to λ is.

$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda .$$

The vector $v^+ = 1 \otimes 1$ is maximal of weight λ, and $M(\lambda) = \langle v^+ \rangle$. So $M(\lambda) \in \mathcal{O}$.

Other ways to define the Verma modules, but let's take this one.

We denote
- $L(\lambda)$: the unique simple quotient of $M(\lambda)$
- $N(\lambda)$ the unique maximal submodule of $M(\lambda)$

Theorem Every simple module of $\mathcal{O}$ is isomorphic to one $L(\lambda)$ and is then determined uniquely by its highest weight. Furthermore $\mathrm{Hom}_{\mathcal{O}}(L(\lambda), L(\mu)) = \delta_{\lambda,\mu}$.

Prop. Frobenius reciprocity.

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M(\lambda), M) \simeq \mathrm{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathbb{C}_\lambda, \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M).$$

Proof from $f : M(\lambda) \longrightarrow M$, use universality of $\otimes$.

$$\begin{tikzcd}
\mathbb{C}_\lambda \arrow[r, dashed, "\exists! f'"] \arrow[d] & \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M \\
\mathcal{U}(\mathfrak{g}) \otimes \arrow[ur, "\mathrm{Res} \circ f"'] &
\end{tikzcd}
\implies f \longmapsto f' \text{ uniquely.}$$

$$f \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M(\lambda), M), \quad f' \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathbb{C}_\lambda, \mathrm{Res}\, M)$$

1.4

This will be investigated later in chapter 4, but still worth to introduce things right now.

— Fix a numbering of simple roots $\alpha_1 \cdots \alpha_l$

Fix a basis $\underbrace{h_1 \cdots h_l}_{\mathfrak{h}}$, $\underbrace{x_1 \cdots x_l}_{\mathfrak{n}^+}$, $\underbrace{y_1, \cdots, y_l}_{\mathfrak{n}^-}$

Lemma Commutation of $\mathcal{U}(\mathfrak{g})$

a) $[x_j, y_i^{k+1}] = 0$ if $j \neq i$

b) $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j) \cdot y_i^{k+1}$

c) $[x_j, y_j^{k+1}] = -(k+1)\, y_j^{k} \, (\underbrace{k \cdot 1 - h_i}_{\in \mathfrak{h}})$

We are searching for maximal vectors of weight different than λ.

Prop Given $\lambda \in \mathfrak{h}^*$ and a simple root α. suppose $n = \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^+$.
If $v^+ \in M(\lambda)$ is a maximal vector of weight λ then
$y_\alpha^{n+1} \cdot v^+$ is a maximal vector of weight $\mu = \lambda - (n+1)\alpha < \lambda$.

Coro This gives a morphism
$$M(\mu) \longrightarrow N(\lambda) \subset M(\lambda)$$

Coro If $v^+ \in L(\lambda)$ instead, then $y_\alpha^{n+1} v^+ = 0$