References

James E. Humphreys
Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$
Graduate Studies in Mathematics **94**
American Mathematical Society, Providence, RI (2008)

James E. Humphreys
Introduction to Lie Algebras and Representation Theory
Springer-Verlag, New York (1972)

Luc Frappat, Antonino Sciarrino, Paul Sorba
Dictionary on Lie algebras and superalgebras.
Academic Press, San Diego, CA (2000)

Wikipedia
some pictures

Lie algebra

Definition.
A vector space L over a field $\mathbb{F}$, with a bracket operation

$$L \times L \to L \colon (x, y) \mapsto [x, y]$$

is called a **Lie algebra** over $\mathbb{F}$ if the following axioms are satisfied:

(L1) The bracket operation is bilinear

(L2) $[x, x] = 0$ for all $x \in L$

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad (x, y, z \in L)$

Notations:
Humphreys: $[xy] = [x, y]$ adjoint repr. $(\operatorname{ad} x)(y) = [x, y]$

Assumptions:
$\mathbb{F}$ algebraically closed field of characteristic 0 such as $\mathbb{C}$
L finite-dimensional

Example

$\mathfrak{sl}(2, \mathbb{C})$ with basis (h, x, y)

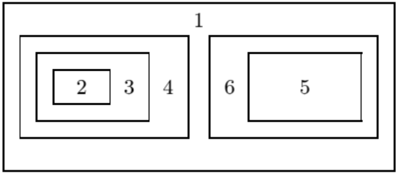
$$[h, x] = 2x \qquad [h, y] = -2y \qquad [x, y] = h$$

matrix realization

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

bracket is commutator

Semisimple Lie algebra



1 Lie Algebras (L.A.)
2 Abelian L.A.
3 Nilpotent L.A.
4 Solvable L.A.
5 Simple L.A.
6 Semi-simple L.A.

arbitrary Lie algebra is semi-direct sum of *solvable* and *semisimple*

L **solvable** if $L^{(n)} = 0$ where $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$

Rad L = **radical** of L, unique maximal solvable ideal

L **semisimple** if Rad $L = 0$ Ex: $L/$ Rad L is semisimple

L **nilpotent** if $L^n = 0$ where $L^i = [L, L^{i-1}]$

Semisimple Lie algebra

Killing form

$$(x, y) := \mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$$

symmetric bilinear form

associative $([x, y], z) = (x, [y, z])$

L is semisimple if and only if Killing form is nondegenerate

$$\{x \in L \mid (x, y) = 0 \text{ for all } y \in L\} = 0$$

semisimple Lie algebras decompose uniquely into a direct sum of simple ideals, which are classified by their (irreducible) root system

$$A_\ell, B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4, G_2$$

Notations

$\mathfrak{g}$ semisimple Lie algebra

$\mathfrak{h} \subset \mathfrak{g}$ Cartan algebra, maximal abelian subalgebra

$\ell = \dim \mathfrak{h}$ rank of $\mathfrak{g}$

for $\alpha \in \mathfrak{h}^*$: $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$

if $\mathfrak{g}_\alpha$ nonzero (and 1-dimensional), then $\alpha \in \mathfrak{h}^*$ **root**

root system $\Phi \subset \mathfrak{h}^*$ all roots

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Root system

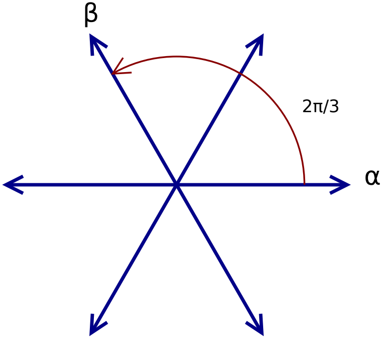
$\Phi^+ \subset \Phi$ positive system $\quad |\Phi^+| = m$

Δ simple system $\quad |\Delta| = \ell$

$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$

$\mathfrak{n}^- = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha \qquad \mathfrak{n} = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$

Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$
maximally solvable



Each $\alpha \in \Phi^+$ determines subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2)$ with basis $(x_\alpha, y_\alpha, h_\alpha)$ normalized such that $h_\alpha = [x_\alpha, y_\alpha]$ and $\alpha(h_\alpha) = 2$

transpose map $\tau : \mathfrak{g} \to \mathfrak{g}$ standard anti-involution, interchanges x_α and y_α for all $\alpha \in \Phi^+$, and fixes $\mathfrak{h}$ pointwise

Extra terminology

coroot $\alpha^\vee := 2\alpha/(\alpha, \alpha)$

Cartan invariant $\langle \beta, \alpha^\vee \rangle := 2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$

dual root system $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$

In the Killing form identification of $\mathfrak{h}$ and $\mathfrak{h}^*$

$$\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha) \quad \text{for all} \beta \in \Phi$$

root lattice Λ_r is $\mathbb{Z}$-span of Φ

simple system Δ forms $\mathbb{Z}$-basis of Λ_r

each $\beta \in \Phi^+$ can be written uniquely, with all $c_\alpha \in \mathbb{Z}^+$, as

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha \qquad \text{the } \textbf{height} \text{ of } \beta \text{ is } \mathrm{ht}\beta = \sum_{\alpha \in \Delta} c_\alpha$$

Weyl Groups

root lattice Λ_r is stable under the action of the **Weyl group** W, the natural symmetry group attached to a root system Φ

finite subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by all reflections $s_\alpha \colon \lambda \to \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for $\alpha \in \Phi$ (or just a fixed simple system Δ)

Coxeter group, satisfies crystallographic restriction

$\ell(w) = \min\{n \mid w = s_1 \cdots s_n \text{ with } s_i \text{ simple reflections } \}$

The number of $\alpha \in \Phi^+$ for which $w\alpha < 0$ is precisely $\ell(w)$

There is a very useful way to partially order W: **Bruhat ordering**

$$w' < w \implies \ell(w') < \ell(w)$$

Integral Weights

integral weight lattice linked to the root system Φ

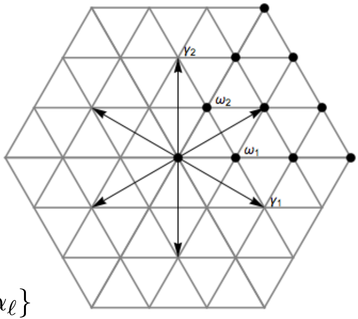
$$\Lambda := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$$

lies in $\mathbb{Q}$-span of the roots

stable under action of W

free abelian group of rank ℓ

includes the root lattice Λ_r
as a subgroup of finite index

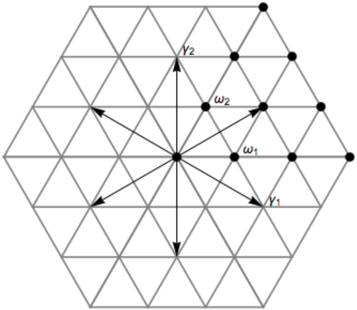


For a fixed simple system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$

natural partial ordering on Λ: $\mu \leq \lambda \iff \lambda - \mu \in \mathbb{Z}^+$-span of Δ

Integral Weights

For a fixed simple system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$



Λ has $\mathbb{Z}$-basis of **fundamental weights**

$\varpi_1, \ldots, \varpi_\ell$ with $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$

dominant integral weights

$$\Lambda^+ := \mathbb{Z}^+\varpi_1 + \cdots + \mathbb{Z}^+\varpi_\ell$$

Given $\lambda \in \Lambda^+$, all $w\lambda \leq \lambda$ for $w \in W$

The number of dominant weights $\leq \lambda$ for a given $\lambda \in \Lambda^+$ is finite

Weyl vector (for all $\alpha \in \Delta$)

$$\rho := \varpi_1 + \cdots + \varpi_\ell = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \qquad \langle \rho, \alpha^\vee \rangle = 1 \qquad s_\alpha \rho = \rho - \alpha$$

Integral Weights

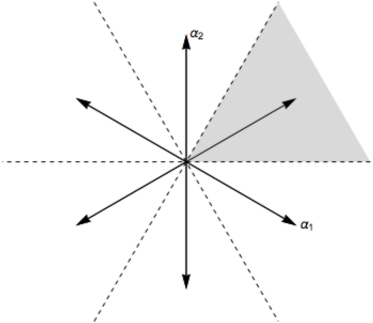
For a fixed simple system Δ

Weyl chamber

$C := \{\lambda \in \mathfrak{h}^* \mid (\lambda, \alpha) > 0 \text{ for all } \alpha \in \Delta\}$

$\overline{C}$ natural fundamental domain for the action of W

natural bijection with simple systems



Lemma for 7.9

$$\lambda \in \overline{C} \iff \lambda \geq s_\alpha \lambda \text{ for all } \alpha \in \Delta \iff \lambda \geq w\lambda \text{ for all } w \in W$$

Universal Enveloping Algebras

$U(L)$ is associative algebra with 1 generated by L

for $x \in L$

$$\operatorname{ad} x \colon U(L) \to U(L) \colon u \mapsto xu - ux$$

noetherian, no zero-divisors, PBW

semisimple Lie algebra $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$

$$y_1^{r_1} \cdots y_m^{r_m} h_1^{s_1} \cdots h_\ell^{s_\ell} x_1^{t_1} \cdots x_m^{t_m}$$

basis element in standard PBW ordering

$$U(\mathfrak{g}) = \bigoplus_{\nu \in \Lambda_r} U(\mathfrak{g})_\nu \qquad U(\mathfrak{g})_\nu = \mathrm{span}\{\text{monomial } \mid \nu = \sum (t_i - r_i)\alpha_i\}$$

Center

$Z(\mathfrak{g})$ center of $U(\mathfrak{g})$

acts by scalars, sometimes even on infinite dimensional modules
(no Schur's Lemma)

Structure: polynomial algebra in ℓ indeterminates (see Chapter 1)

Casimir: special element $Z(\mathfrak{g})$ using nondegeneracy Killing form

for $\mathfrak{g} = \mathfrak{sl}(2)$: $h^2 + 2xy + 2yx = h^2 + 2h + 4yx$

τ extends to anti-automorphism of $U(\mathfrak{g})$
fixes $Z(\mathfrak{g})$ pointwise (using properties of the Harish-Chandra homomorphism, see Chapter 1)

Representations

representations of $\mathfrak{g}$ or $U(\mathfrak{g})$, not necessarily finite dimensional

category $\operatorname{Mod} U(\mathfrak{g})$ of all (left) $U(\mathfrak{g})$-modules

$\mathcal{O}$ is well-behaved subcategory of $\operatorname{Mod} U(\mathfrak{g})$ (see Chapter 1)

$\lambda \in \mathfrak{h}^*$ is a **weight** of $M \in \operatorname{Mod} U(\mathfrak{g})$ if **weight space** relative to $\mathfrak{h}$

$$M_\lambda := \{v \in M \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\} \neq 0$$

multiplicity of λ in M is $\dim M_\lambda$

notation $\Pi(M) := \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$

M is a **weight module** if it is direct sum of its weight spaces

i.e. $\mathfrak{h}$ acts semisimply on M

Finite Dimensional Modules

Weyl's Complete Reducibility Theorem

Every finite dimensional $U(\mathfrak{g})$-module is isomorphic to a direct sum of simple modules, with uniquely determined multiplicities.

When $\dim M < \infty$, M is always a weight module

elements of $\mathfrak{h}$ act via semisimple matrices

elements of $\mathfrak{n}$ or $\mathfrak{n}^-$ act via nilpotent matrices

The set $\Pi(M)$ of weights is W-invariant, with $\dim M_\lambda = \dim M_{w\lambda}$

All weights of M are integral

Simple finite dim. modules for $\mathfrak{sl}(2)$

$\mathfrak{sl}(2,\mathbb{C})$ with basis (h,x,y)

$$[h,x]=2x \qquad [h,y]=-2y \qquad [x,y]=h$$

weights $\lambda\in\mathfrak{h}^*\leftrightarrow\mathbb{C}$

root lattice $\Lambda_r\leftrightarrow 2\mathbb{Z}$

integral weight lattice $\Lambda\leftrightarrow\mathbb{Z}$

simple modules $L(\lambda)\leftrightarrow\lambda\in\Lambda^+$ dominant integral weights

basis $v_0,\ldots,v_\lambda$ and set $v_{-1}=0=v_{\lambda+1}$

$$\begin{aligned} h\cdot v_i &= (\lambda-2i)v_i \\ x\cdot v_i &= (\lambda-i+1)v_{i-1} \\ y\cdot v_i &= (i+1)v_{i+1} \end{aligned}$$