Kleine Seminar 20/10/2020.

Goal To categorify Fock representation of the Heisenberg algebra.

|Lie algebra|

H : $\cancel{\{x,y,z\}}$

$\langle x, y, z \rangle \qquad [x, y] = \langle x, y \rangle z$

$H_{2n+1} \ \langle x_i, y_i, z \rangle$: $\begin{cases} [x_i, y_j] = \langle x_i, y_j \rangle z \\ [x_i, x_j] = 0 \\ [y_i, y_j] = 0 \end{cases}$

central element.

$$\Rightarrow \left\{ \begin{pmatrix} 0 & \vec{x} & z \\ & \ddots & \vec{y} \\ & & 0 \end{pmatrix} \right.$$

$x = q$

$y = i.p$

Today: Infinite-dimensional, unital associative algebra over $\mathbb{Q}$. H^{q}_{∞}

$\left.\begin{array}{l} p_i,\ i \in \mathbb{N}_+,\ q_j,\ j \in \mathbb{N}_+ \\ [p_i, p_j] = [q_i, q_j] = 0 \\ [p_i, q_j] = i\,\delta_{ij} \end{array}\right\} \Rightarrow$ not suited for categorification.

Consider integer Heisenberg algebra H over $\mathbb{Z}$ (unital, associative),

$e_n,\ h^*_m,\ n, m \in \mathbb{N}_+$

s.t.

$[e_n, e_m] = [h^*_n, h^*_m] = 0$

$[h^*_m, e_n] = e_{n-1} h^*_{m-1} \Leftrightarrow h^*_m e_n = e_n h^*_m + e_{n-1} h^*_{m-1}$

$H \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^{q}_{\infty}$

|Realisation using $\mathcal{S}$-ym|

<u>Outline</u>

- Introduce Sym
- Define several bases • $H \subseteq \mathrm{End}_{\mathbb{Z}}(\mathrm{Sym}) \rightsquigarrow$ Fock representation
- Introduce $\mathcal{A}$ category of symmetric group modules.

$$\Rightarrow \mathcal{K}(\mathcal{A}) \cong \mathrm{Sym}$$

- For each $M \in \mathrm{Obj}(\mathcal{A})$

$$\mathrm{Res}_M : \mathcal{A} \to \mathcal{A}$$
$$\mathrm{Ind}_M : \mathcal{A} \to \mathcal{A}$$
functors.

$\longrightarrow$ weak categorification.

Sym : algebra ~~over~~ of symmetric ~~functions~~ polynomials? in countable many variables over $\mathbb{Z}$.

<u>In 2 variables</u> $f(x,y) = f(y,x)$

<u>Ex</u> $x+y$, x^2+y^2, xy,

$$\mathrm{Sym} = \bigoplus_{n \in \mathbb{N}} \mathrm{Sym}_n$$
homogeneous polynomials of degree n.

$\mathrm{Sym}_0 = \mathbb{Z}\, 1$ $\quad \boxed{m_\varnothing}$

$\mathrm{Sym}_1 = \mathbb{Z}(x_1 + x_2 + \cdots) = \mathbb{Z}\, S(x_1)$ $\quad \boxed{m_{\square}}$

$\mathrm{Sym}_2 = \mathbb{Z}(x_1^2 + x_2^2 + \cdots) + \mathbb{Z}(x_1x_2 + x_1x_3 + x_2x_3 + \cdots) = \mathbb{Z}\, S(x_1^2) + \mathbb{Z}(x_1x_2)$ $\quad \boxed{m_{\square\square}}$ $\quad \boxed{m_{\boxminus}}$

$\mathrm{Sym}_3 = \mathbb{Z}\, S(x_1^3) + \mathbb{Z}\, S(x_1^2 x_2) + \mathbb{Z}\, S(x_1x_2x_3)$

$m_{\square\square\square}$ $\quad m_{\square\square,\square}$ $\quad m_{\square,\square,\square}$

Basis labelled by partitions:

$\lambda \in \mathcal{P}$

$$m_\lambda = \sum_{\substack{\alpha \\ \text{rearrangement} \\ \text{of } \lambda}} x^\alpha \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_i^{\alpha_i} \cdots$$

$\boxed{\text{monomial}}$

$\{m_\lambda \mid \lambda \in \mathcal{P}\}$ is a $\mathbb{Z}$-basis.

→ complete symmetric function.

$$h_n = \sum_{\lambda \in \mathcal{P}_n} m_\lambda \qquad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_q}$$

→ elementary

$$e_n = m_{(1^n)} \qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_q}$$

→ power sum

$$p_n = m_{(n)} \qquad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_q}.$$

Ex 1

$h_0 = e_0 = p_0 = m_\emptyset = 1$

$h_1 = e_1 = p_1 = m_{\square} = x_1 + x_2 + \dots$

$h_2 = S(x_1^2) + S(x_1 x_2) = h_{\square\square}$

$e_2 = S(x_1 x_2) = e_{\square\square}$

$p_2 = S(x_1^2) = p_{\square\square}$

$h_3 = S(x_1^3) + S(x_1^2 x_2) + S(x_1 x_2 x_3)$

$e_3 = S(x_1 x_2 x_3)$

$p_3 = S(x_1^3)$

$h_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = S(x_1)\,S(x_1) = x_1^2 + x_1 x_i + x_i x_1 = S(x_1^2) + 2S(x_1 x_2)$

$\parallel$

$e_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = p_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$

$h_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = h_2 \cdot h_1 = S(x_1^3) + 3S(x_1^2 x_2)$

$h_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = S(x_1^3) + 5S(x_1^2 x_3) + 2S(x_1 x_2 x_3)$

$e_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = S(x_1 x_2 x_3) + S(x_1^2 x_2)$

$e_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = S(x_1^3) + 5S(x_1^2 x_3) + 2S(x_1 x_2 x_3)$

$p_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = S(x_1^2) \cdot S(x_1) = S(x_1^3) + S(x_1^2 x_2)$

$p_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = e_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = h_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$

$\Rightarrow \left\{\begin{matrix} h_\lambda \\ e_\lambda \end{matrix}\right. \ \lambda \in \mathcal{P}$ are $\mathbb{Z}$-bases for Sym $\Big|$ $\{p_\lambda\}_{\lambda \in \mathcal{P}}$ only $\mathbb{Q}$-bases.

Schur functions |

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le n} \quad \lambda \in P.$$

$n >$ length of λ.

$$s_\emptyset = h_0 = 1$$

$$s_1 = \begin{matrix} \cancel{h_1} & h_2 \\ \cancel{h_{-1}} & h_0 \end{matrix} \quad \begin{vmatrix} h_1 & h_2 \\ h_{-1} & h_0 \end{vmatrix} = \begin{vmatrix} h_1 & h_2 \\ 0 & 1 \end{vmatrix} = h_1$$

$$s_{\square\square} = \begin{vmatrix} h_2 & h_3 \\ h_{-1} & h_0 \end{vmatrix} = h_2$$

$$s_{\substack{\square \\ \square}} = \begin{vmatrix} h_1 & h_2 \\ h_0 & h_1 \end{vmatrix} = h_1^2 - h_2 = -2S(x_1 x_2) \overset{?}{=} e_2$$

$$s_{\square\square\square} = \begin{vmatrix} h_3 & h_4 & h_5 \\ h_{-1} & h_0 & h_1 \\ h_{-2} & h_{-1} & h_0 \end{vmatrix} = h_3$$

$$s_{(n)} = h_n.$$

$$s_{\substack{\square \\ \square \\ \square}} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 \\ h_0 & h_1 & h_2 & h_3 \\ h_{-1} & h_0 & h_1 & h_2 \\ h_{-3} & h_{-2} & h_{-1} & h_0 \end{pmatrix} = \begin{vmatrix} h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix} = h_1^3 + h_3 - 2h_1 h_2$$

$$= S(x_1^3) + 5S(x_1^2 x_2) + 2S(x_1 x_2 x_3)$$
$$+ S(x_1^3) + S(x_1^2 x_2) + S(x_1 x_2 x_3)$$
$$- 2S(x_1^3) - 6S(x_1^2 x_2) = 3S(x_1 x_2 x_3) \overset{?}{=} e_3$$

Define inner product |

$$\begin{cases} \langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}. \\ \Rightarrow \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}. \end{cases}$$

Heisenberg

Heisenberg algebra.

Let $f \in Sym$ act on $Sym_{\mathbb{Q}}$.

$$f: x \mapsto f \cdot x.$$

f^* adjoint.

$f^*(x)$ is s.t. that.

$$< f(b), a > = < b, f^* a >$$

Well-defined since $<\cdot,\cdot>$ is non-degenerated.

Algebra generated by f, f^* is the Heisenberg algebra.

e_n generators for Sym

h_n^* generators for Sym^*

$\to$ satisfy the relations of the Heisenberg algebra.

and $H \cong Sym \otimes Sym^*$

Category $\mathcal{A}$

$A_n = \mathbb{C}[S_n]$ (semisimple)

irreps: labelled by $\mathcal{P}_n$

$S^\lambda \to$ specht-module.

$E_n = S^{(1^n)} \to$ sign representation

$L_n = S^{(n)} \to$ trivial representation.

$$\mathcal{A} = \bigoplus_{n \in \mathbb{N}} A_n\text{-mod}$$

$$G_{\mathcal{A}} = \bigoplus_{n \in \mathbb{N}} G_0(A_n) = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{Z}[S^\lambda]$$

Bilinear form on $G_{\mathcal{A}}$ (well-defined since $G_0(A_n) = K_0(A_n)$)

$$<\cdot,\cdot>: G_{\mathcal{A}} \times G_A \to \mathbb{Z}$$

$$[M] \times [N] \mapsto \dim_{\mathbb{C}} \mathrm{Hom}_A(M,N)$$

Proposition

$\varphi_A : G_A \to Sym$

$[S^\lambda] \mapsto s_\lambda$ for all $\lambda \in P$

is an isomorphism.

$[E_n] \mapsto e_n$

$[L_n] \mapsto h_n$

Furthermore

$\langle a, b \rangle = \langle \varphi_A(a), \varphi_A(b) \rangle$ for all $a, b \in G_A$.

Let $M \in \mathcal{A}$. , $M \in A_m$ $A_m \otimes A_n \hookrightarrow A_{m+n}$

Then

$\mathrm{Ind}_M : \mathcal{A} \to \mathcal{A}$.

$N \mapsto \mathrm{Ind}_{A_m \otimes A_n}^{A_{m+n}} \underbrace{M \otimes N}_{\in A_m \otimes A_n}$. $A_{m+n} \otimes_{A_m \otimes A_n} M \otimes N$.

$\underset{A_n}{\overset{\Uparrow}{N}}$

$\mathrm{Res}_M : \mathcal{A} \to \mathcal{A}$.

$m \leq n$ $\underset{A_n}{\overset{\Uparrow}{N}} \mapsto \underset{A_{n-m}}{\Uparrow}\mathrm{Hom}_{A_m}(M, \mathrm{Res}_{A_{n-m} \otimes A_m}^{A_n} N)$

$m > n$ $N \mapsto 0$

If L is an $A_k \otimes A_\ell$ module, M an A_ℓ-module Then $\mathrm{Hom}_{A_\ell}(M, L)$ is an A_k module by. $(a \cdot f)(m) = (1 \otimes a) f(m)$

$\Rightarrow$ Exact functors. Thus induce operators on $G_{\mathcal{A}}$.

Prop

$$
\begin{tikzcd}
G_A \arrow[r, "{[\mathrm{Ind}_M]}"] \arrow[d, "\varphi_A"'] & G_A \arrow[d, "\varphi_A"] \\
\mathrm{Sym} \arrow[r, "{\varphi_A[M]}"] & \mathrm{Sym}
\end{tikzcd}
\qquad
\begin{tikzcd}
G_A \arrow[r, "{[\mathrm{Res}_M]}"] \arrow[d, "\varphi_A"'] & G_A \arrow[d, "\varphi_A"] \\
\mathrm{Sym} \arrow[r, "{(\varphi_A(M))^*}"] & \mathrm{Sym}
\end{tikzcd}
$$

Prop

$$\mathrm{Ind}_{E_n} \circ \mathrm{Ind}_{E_m} \cong \mathrm{Ind}_{E_m} \circ \mathrm{Ind}_{E_n}$$

$$\mathrm{Res}\, L_n \circ \mathrm{Res}\, L_m \cong \mathrm{Res}\, L_m \circ \mathrm{Res}\, L_n.$$

$$\mathrm{Res}\, L_m \circ \mathrm{Ind}_{E_n} \cong \mathrm{Ind}_{E_n} \circ \mathrm{Res}\, L_m \oplus \mathrm{Ind}_{E_{n-1}} \circ \mathrm{Res}\, L_{m-1}.$$