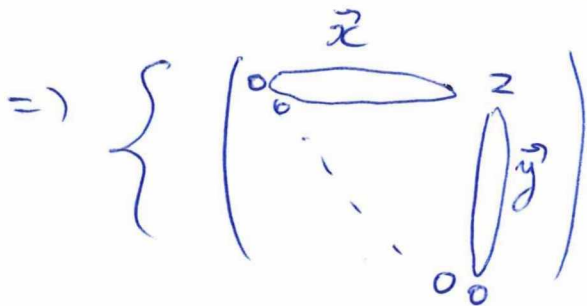


Goal To categorify Fock representation of the Heisenberg algebra.

$\mathfrak{H} : \langle x, y, z \rangle$ Lie algebra
 $[x, y] = \langle x, y \rangle z$

$\mathfrak{H}_{m+1} \langle x_i, y_i, z \rangle$:
 central element. $\left\{ \begin{array}{l} [x_i, y_j] = \langle x_i, y_j \rangle z \\ [x_i, x_j] = 0 \\ [y_i, y_j] = 0 \end{array} \right.$



$x = \mathbb{R}$
 $y = \mathbb{R} \cdot p$

Today: Infinite-dimensional, unital associative algebra over \mathbb{Q} .

$\left. \begin{array}{l} p_i, i \in \mathbb{N}_+, q_j, j \in \mathbb{N}_+ \\ [p_i, p_j] = [q_i, q_j] = 0 \\ [p_i, q_j] = \delta_{ij} \end{array} \right\} \Rightarrow$ not suited for categorification.

Consider integer Heisenberg algebra \mathfrak{H} over \mathbb{Z} . (unital, associative)

$e_n, h_m, n, m \in \mathbb{N}_+$

r.c.

$[e_n, e_m] = [h_n^*, h_m^*] = 0$

$[h_m^*, e_n] = e_{n-1} h_{m-1}^* \Leftrightarrow h_m^* e_n = e_n h_m^* + e_{n-1} h_{m-1}^*$

$\mathfrak{H} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathfrak{H}_{\infty}$

Realisation ~~is~~ using Sym.

Outline

- Introduce Sym
- Define several bases
- $H \in \text{End}_{\mathbb{Z}}(\text{Sym}) \rightarrow$ Frod representation
- Introduce a category of \mathbb{Z} symmetric group modules.

$\Rightarrow \mathcal{K}(U) \cong \text{Sym}$

- For each $M \in \text{Obj}(U)$

$\text{Res}_n : U \rightarrow U$ functors.

$\text{Ind}_n : U \rightarrow U$

\rightarrow weak categorification.

Sym : algebra over \mathbb{Z} of symmetric functions in countable many variables over \mathbb{Z} . *polynomials?*

In 2 variables | $f(x,y) = f(y,x)$

ex $x+y, x^2+y^2, xy,$

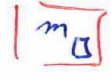
homogeneous polynomials of degree n.

$\text{Sym} = \bigoplus_{n \in \mathbb{N}} \text{Sym}_n$

$\text{Sym}_0 = \mathbb{Z} 1$



$\text{Sym}_1 = \mathbb{Z}(x_1 + x_2 + \dots) = \mathbb{Z} S(x_1)$



$\text{Sym}_2 = \mathbb{Z}(x_1^2 + x_2^2 + \dots) + \mathbb{Z}(x_1 x_2 + x_1 x_3 + x_2 x_3 + \dots)$



$\text{Sym}_3 = \mathbb{Z} S(x_1^3) + \mathbb{Z} S(x_1^2 x_2) + \mathbb{Z} S(x_1 x_2 x_3)$



Basis labelled by partitions:

$\lambda \in \mathcal{P}$

$m_\lambda = \sum_{\alpha} x^\alpha$
rearrangement of λ .

$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_i^{\alpha_i} \dots$

monomial

$\{m_\lambda \mid \lambda \in \mathcal{P}\}$ is a \mathbb{Z} -basis.

→ complete symmetric function.

$$h_m = \sum_{\lambda \in \mathcal{O}_m} m_\lambda \quad h_\lambda = a_{\lambda_1} a_{\lambda_2} \dots a_{\lambda_q}$$

→ elementary

$$e_m = m_{(1^m)} \quad e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_q}$$

→ Power sum

$$p_m = m_{(m)} \quad p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_q}$$

Ex 1

$$h_0 = e_0 = p_0 = m_\emptyset = 1$$

$$a_3 = S(x_1^3) + S(x_1^2 x_2) + S(x_1 x_2 x_3)$$

$$h_1 = e_1 = p_1 = m_{\mathbb{H}} = x_1 + x_2 + \dots$$

$$e_3 = S(x_1 x_2 x_3)$$

$$h_2 = S(x_1^2) + S(x_1 x_2) = a_{\mathbb{H}\mathbb{H}}$$

$$p_3 = S(x_1^3)$$

$$e_2 = S(x_1 x_2) = e_{\mathbb{H}\mathbb{H}}$$

$$p_2 = S(x_1^2) = p_{\mathbb{H}\mathbb{H}}$$

$$h_{\mathbb{H}\mathbb{H}} = S(x_1) S(x_1) = x_1^2 + x_1 x_2 + x_2 x_1 = S(x_1^2) + 2S(x_1 x_2)$$

$$\parallel$$

$$e_{\mathbb{H}\mathbb{H}} = p_{\mathbb{H}\mathbb{H}}$$

$$h_{\mathbb{H}\mathbb{H}\mathbb{H}} = h_2 \cdot h_1 = S(x_1^3) + 3S(x_1^2 x_2)$$

$$a_{\mathbb{H}\mathbb{H}\mathbb{H}} = S(x_1^3) + 5S(x_1^2 x_2) + 2S(x_1 x_2 x_3)$$

$$e_{\mathbb{H}\mathbb{H}\mathbb{H}} = S(x_1 x_2 x_3) + S(x_1^2 x_2)$$

$$e_{\mathbb{H}\mathbb{H}} = S(x_1^3) + 5S(x_1^2 x_2) + 2S(x_1 x_2 x_3)$$

$$p_{\mathbb{H}\mathbb{H}} = S(x_1^2) \cdot S(x_1) = S(x_1^3) + S(x_1^2 x_2)$$

$$p_{\mathbb{H}\mathbb{H}} = e_{\mathbb{H}\mathbb{H}} = a_{\mathbb{H}\mathbb{H}}$$

⇒ $\begin{cases} h_\lambda \\ e_\lambda \end{cases} \lambda \in \mathcal{O}$ are \mathbb{Z} -bases for sym. $\left\{ \begin{matrix} S \\ P_\lambda \end{matrix} \right\}_{\lambda \in \mathcal{O}}$ only \mathbb{Q} -bases.

S-char functions

4

$$\tau_\lambda = \det (a_{\lambda_i - i + j})_{1 \leq i, j \leq n}, \quad \lambda \in \mathcal{O}.$$

 $n > \text{length of } \lambda.$

$$\tau_\emptyset = a_0 = 1$$

$$\tau_1 = \begin{vmatrix} a_1 & a_2 \\ a_0 & a_0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ 0 & 1 \end{vmatrix} = a_1$$

$$\tau_{\square} = \begin{vmatrix} a_2 & a_3 \\ a_1 & a_0 \end{vmatrix} = a_2$$

$$\tau_H = \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} = a_1^2 - a_2 = -2S(\lambda_1 \lambda_2) \stackrel{?}{=} e_2$$

$$\tau_{\square\square} = \begin{vmatrix} a_3 & a_4 & a_5 \\ a_1 & a_0 & a_1 \\ a_2 & a_1 & a_0 \end{vmatrix} = a_3$$

$$\tau_{(n)} = a_n.$$

$$\tau_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ a_{-1} & a_0 & a_1 & a_2 \\ a_{-2} & a_{-1} & a_0 & a_1 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ 1 & a_1 & a_2 \\ 0 & 1 & a_1 \end{vmatrix} = a_1^3 + a_3 - 2a_1 a_2$$

$$= S(\lambda_1^3) + 5S(\lambda_1^2 \lambda_2) + 2S(\lambda_1 \lambda_2 \lambda_3)$$

$$+ S(\lambda_1^3) + S(\lambda_1^2 \lambda_2) + S(\lambda_1 \lambda_2 \lambda_3)$$

$$- 2S(\lambda_1^3) - 6S(\lambda_1^2 \lambda_2) = 3S(\lambda_1 \lambda_2 \lambda_3) \stackrel{?}{=} e_3$$

before inner product

$$\left\{ \begin{array}{l} \langle m_\lambda, a_\mu \rangle = \delta_{\lambda\mu}. \\ \Rightarrow \langle \tau_\lambda, \tau_\mu \rangle = \delta_{\lambda\mu}. \end{array} \right.$$

Hausberg

Heisenberg algebra.

5

Let $f \in \text{Sym}$ act on $\text{Sym} \mathbb{C}^n$.

$$f: \mathbb{C}^n \mapsto f \cdot \mathbb{C}^n.$$

f^* adjoint.

$f^*(\mathbb{C}^n)$ is s.c. dual.

$$\langle f(\mathbb{C}^n), \alpha \rangle = \langle \mathbb{C}^n, f^* \alpha \rangle.$$

Well-defined, since $\langle \cdot, \cdot \rangle$ is non-degenerated.

Algebra generated by f, f^* is the Heisenberg algebra.

e_n generators for Sym

e_n^* generators for Sym^*

\rightarrow satisfy the relations of the Heisenberg algebra.

$$\text{and } H \cong \text{Sym} \otimes \text{Sym}^*$$

Category \mathcal{A}

$$A_n = \mathbb{C}[S_n] \quad (\text{semisimple})$$

irreps: labelled by \mathcal{O}_n

$S^\lambda \rightarrow$ Specht-module.

$$E_n = S^{(1^n)} \rightarrow \text{sign representation}$$

$$L_n = S^{(n)} \rightarrow \text{trivial representation.}$$

$$\mathcal{A} = \bigoplus_{n \in \mathbb{N}} A_n\text{-mod}$$

$$G_{\mathcal{A}} = \bigoplus_{n \in \mathbb{N}} G_0(A_n) = \bigoplus_{\lambda \in \mathcal{O}} \mathbb{Z}[S^\lambda]$$

Bilinear form on $G_{\mathcal{A}}$ (well-defined since $G_0(A_n) = K_0(A_n)$)

$$\langle \cdot, \cdot \rangle: G_{\mathcal{A}} \times G_{\mathcal{A}} \rightarrow \mathbb{Z}$$

$$[M] \times [N] \mapsto \dim_{\mathbb{C}} \text{Hom}_A(M, N)$$

Proposition

$$\varphi_A: G_A \rightarrow \text{Sym}$$

$$[S^\lambda] \mapsto \varphi_\lambda \quad \text{for all } \lambda \in \mathcal{P}$$

is an isomorphism.

$$[E_n] \mapsto e_n$$

$$[L_n] \mapsto a_n$$

Furthermore

$$\langle a, b \rangle = \langle \varphi_A(a), \varphi_A(b) \rangle \quad \text{for all } a, b \in G_A.$$

Let $M \in \mathcal{A}$, $M \in A_m$

$$A_m \otimes A_n \hookrightarrow A_{m+n}$$

Then

$$\text{Ind}_M: \mathcal{A} \rightarrow \mathcal{A}$$

$$N \mapsto \text{Ind}_{A_m \otimes A_n}^{A_{m+n}} M \otimes N$$

\uparrow
 A_m

$$A_{m+n} \otimes_{A_m \otimes A_n} M \otimes N$$

$$\text{Res}_M: \mathcal{A} \rightarrow \mathcal{A}$$

$$N \mapsto \text{Hom}_{A_m}(M, \text{Res}_{A_m \otimes A_n}^{A_{m+n}} N)$$

\uparrow
 A_{m+n}

$$m > n \quad N \mapsto 0$$

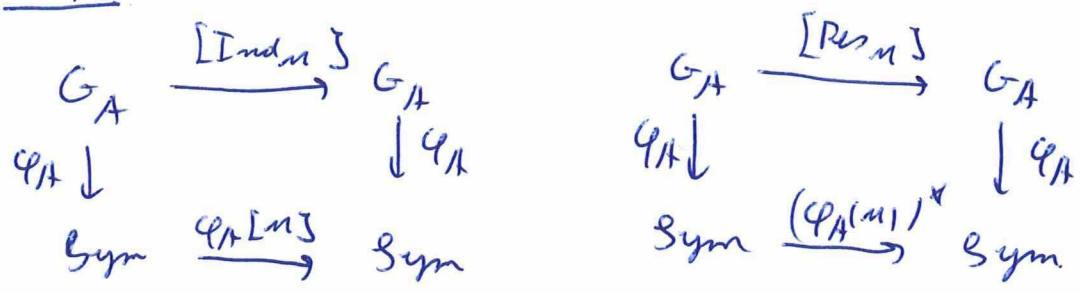
If L is an $A_n \otimes A_n$ module, M an A_n -module

Then $\text{Hom}_{A_n}(M, L)$ is an A_n module by

$$(\alpha \cdot f)(m) = f(1 \otimes \alpha)(m)$$

\Rightarrow Exact functor. Thus induce operators on $G_{\mathcal{A}}$.

Proof



Proof

$$\text{Ind}_{E_n} \circ \text{Ind}_{E_m} \cong \text{Ind}_{E_m} \circ \text{Ind}_{E_n}$$

$$\text{Res}_{L_n} \circ \text{Res}_{L_m} \cong \text{Res}_{L_m} \circ \text{Res}_{L_n}$$

$$\text{Res}_{L_m} \circ \text{Ind}_{E_n} \cong \text{Ind}_{E_n} \circ \text{Res}_{L_m} \oplus \text{Ind}_{E_{n-1}} \circ \text{Res}_{L_{m-1}}$$